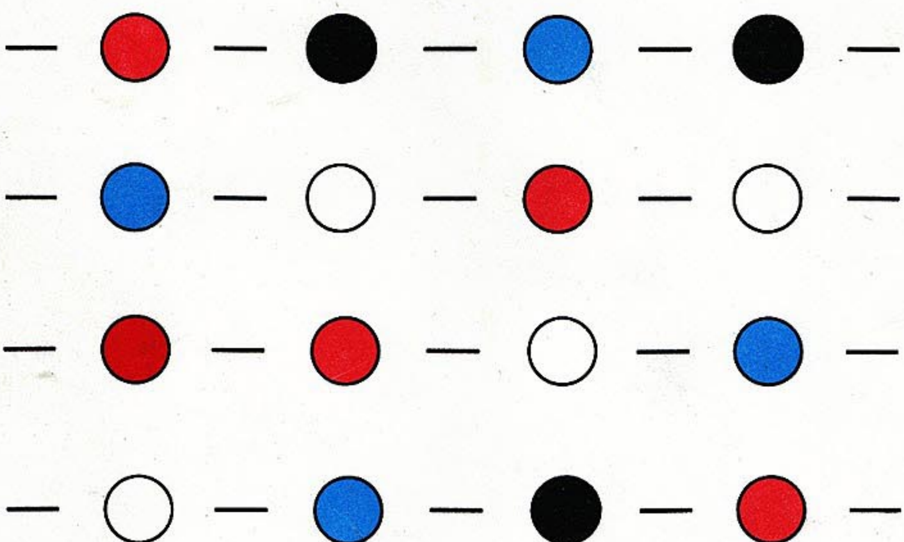
COHERENT STRUCTURES
AND
DECISIONS

by
PANTELIS M. PECHLIVANIDES



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Prologos

The aggregation of individual wishes to social wishes has intrigued human thought for many years because of its close relation to government forms in human societies.

This book is not trying to present the contributions already made in this area. The problem is approached from a new view point: The use of coherent structures (or monotone boolean functions) also used in reliability theory of systems of operating components.

Minimum paths and minimum cuts of structures and their properties lead to definitions of classes of coherent structures so that any p minimum paths (cuts) have at least one common component. These classes of structures when associated with components that respect certain logic properties for logic conjunction or disjunction of statements (alternatives) can guarantee consistent answers up to certain level of conjunction (or disjunction) of statements or silence. Self dual structures to which odd majority structures also belong are shown to be the intersection of the above classes of structures.

Structural properties are investigated to show that symmetric k out of n structures can lead to other structures by contraction of components (formation of parties) and omission of minimum paths. Inversely, it is shown that any coherent structure can be

reached from a k out of n structure by the operations of omission of paths and contraction of components.

Families of symmetric k out of n structures are determined that can guarantee both the existence of answers and logically consistent answers up to certain level of conjunction or disjunction of statements. This result is extended to families of structures that are the result of contraction from k out of n structures which resemble the formation of parties in decision making bodies.

The use of probability theory permits us to find the probability for a structure passing or rejecting statements, as well as the probability of its being inconsistent. Connection is made to other theories like: utility theory, multilinear utility functions and to Arrow's conditions.

Abstentions are dealt as changes to the form of the coherent structure. The passing or blocking power of a component or a group of components is defined and related to the Shapley value of a game. Conditions on preserving relative passing and blocking power lead to certain forms of abstentions for symmetric structures.

Finally, hierarchies of coherent structures are dealt as algorithms that also demonstrate the relation to well known electoral systems.

P. M. P.

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BOOK 1

.COMPONENTS STRUCTURES AND LOGIC

Chapter 1

STRUCTURES

1.1 Some Classes of Coherent Structures: $\mathcal{S}_p$, $\overline{\mathcal{S}}_p$, $\mathcal{M}$

Coherent structures (also known as monotone boolean functions) play an important role in reliability theory; but as it will appear later in this work, they are essential in the study of aggregation of individual preferences. For this reason we will study some classes of coherent structures that will help us understand better the mechanisms of aggregating individual logic. We will be rather brief in our exposition of the notion of coherent structures. The interested reader is referred to the book by R. Barlow and F. Proschan (1975) on reliability theory.

Notation

Let $\varphi(\mathbf{x})$ be a coherent structure with $\mathbf{x}=x_1, \ldots, x_n$ where for all i

$$x_i = \begin{cases} 1 & \text{if component } i \text{ is working} \\ 0 & \text{otherwise} \end{cases}$$

$$\varphi(\mathbf{x}) = \begin{cases} 1 & \text{if the structure is working or passes or accepts } \mathbf{x} \\ 0 & \text{otherwise} \end{cases}$$

By definition of coherent structure we assume that

(a) $\varphi(\mathbf{x})$ is nondecreasing in each x_i

(b) $\varphi(\mathbf{1})=1$

(c) $\varphi(\mathbf{0}) = 0$

where $\varphi(\mathbf{1}) \equiv \varphi(1, \ldots, 1)$ and $\varphi(\mathbf{0}) \equiv \varphi(0, \ldots, 0)$

Finally, all components of φ are *relevant* i.e. for each i

$\varphi(1_i, \mathbf{x}) - \varphi(0_i, \mathbf{x}) = 1$ for some $\mathbf{x}$, where

$\varphi(1_i, \mathbf{x}) = \varphi(x_1, \ldots, x_{i-1}, 1_i, x_{i+1}, \ldots, x_n)$ and

$\varphi(0_i, \mathbf{x}) = \varphi(x_1, \ldots, x_{i-1}, 0_i, x_{i+1}, \ldots, x_n)$

It is also known that a coherent structure can be represented by its min paths or its min cuts. Further, the dual structure of $\varphi(\mathbf{x})$ written as $\varphi^D(\mathbf{x})$ is defined as,

$$\varphi^D(\mathbf{x}) \equiv 1 - \varphi(\mathbf{1} - \mathbf{x})$$

where $(\mathbf{1} - \mathbf{x}) \equiv (1 - x_1, \ldots, 1 - x_n)$.

It can be shown that,

(a) the dual of the dual is the original structure.

(b) the minimal cut (path) sets of the structure are the minimal path (cut) sets of the dual.

(c) If $\varphi^D(\mathbf{x}) \equiv \varphi(\mathbf{x})$ for all $\mathbf{x}$, it follows that each min path set of $\varphi(\mathbf{x})$ is also a min cut set of $\varphi(\mathbf{x})$

Any coherent structure can be expanded in the following form,

$$\varphi(\mathbf{x}) = x_i \varphi(1_i, \mathbf{x}) + (1 - x_i) \varphi(0_i, \mathbf{x})$$

Repeated application of the above decomposition leads to,

$$\varphi(\mathbf{x}) = \sum_{\mathbf{y}} \prod_{j=1}^{n} x_j^{y_j} (1 - x_j)^{1 - y_j} \varphi(\mathbf{y})$$

A useful family of coherent structures is the family of

symmetric structures also called *k out of n* structures.

Their structure function φ has the property that

$$\varphi(\mathbf{x}) = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} x_i \geq k \\ 0 & \text{otherwise} \end{cases}$$

They consist of min paths with k components each and there are $\binom{n}{k}$ min paths in the structure. Also the dual of a k out of n structure is a n-k+1 out of n structure because min paths replace min cuts.

Let $\mathcal{C}$ be the class of all coherent structures. We will also define the following classes,

Definition

$\mathcal{S}_2$ is the subset of coherent structures containing those structures whose min paths satisfy the property that any two min paths within the same structure have at least one common component.

Note: The definition immediately implies that any min path of $\varphi \in \mathcal{S}_2$ is also a cut but not necessarily a min cut.

Definition

$\overline{\mathcal{S}}_2$ is the subset of coherent structures whose min paths satisfy the property that any two min cuts within the same structure have at least one common component.

Note: The definition implies that any min cut of $\varphi \in \overline{\mathcal{S}}_2$ is also a path but not necessarily a min path.

It now follows that $\varphi \in \mathcal{S}_2 \iff \varphi^D \in \overline{\mathcal{S}}_2$ because the min paths of φ are the min cuts of φ^D.

Example 1.1

Take a symmetric structure φ (k out of n) then,

(a) If $k \geq (n+1)/2$ then $\varphi \in \mathcal{S}_2$

(b) If $k \leq (n+1)/2$ then $\varphi \in \overline{\mathcal{S}}_2$

(c) If $k = (n+1)/2$ then $\varphi \in \boldsymbol{M}$

To show (a) we observe that if to the contrary there were two paths with no common component, it would mean that φ has a total of no less than $2(n+1)/2$ different components. But this exceeds the total number of components which is n. Contradiction.

Now define

$$\rho_C(\mathbf{x}) = \varphi(\mathbf{x})\,\varphi(\mathbf{1}-\mathbf{x})$$

and

$$\rho_B(\mathbf{x}) = \varphi^D(\mathbf{x})\,\varphi^D(\mathbf{1}-\mathbf{x})$$

Then we obtain two equivalent definitions of $\mathcal{S}_2$ and $\overline{\mathcal{S}}_2$ respectively,

Proposition 1.2

$$\mathcal{S}_2 = \{\varphi(\mathbf{x}) \mid \rho_C(\mathbf{x}) = 0, \text{ for all } \mathbf{x}\}$$

Proof:

(i) Take $\varphi(\mathbf{x})$ so that $\rho_C(\mathbf{x}) = 0$. Suppose, to the contrary, that $\varphi(\mathbf{x})$ has two min paths with no common component. Then by

choice of $\mathbf{x}$, I can make all components in the one path equal to 1 and all components in the other path equal to 0. Then $\varphi(\mathbf{x})=1$ and $\varphi(\mathbf{1}-\mathbf{x})=1$. Therefore $\rho_C(\mathbf{x})=1$ for the chosen $\mathbf{x}$. Contradiction.

(ii) Take $\varphi(\mathbf{x})$ so that any two min paths have at least one common component. Suppose, to the contrary, that for some $\mathbf{x}$, $\rho_C(\mathbf{x})=1$. This implies $\varphi(\mathbf{x})=1$ and $\varphi(\mathbf{1}-\mathbf{x})=1$, which in turn implies that $\varphi(\mathbf{x})$ has a min path with all components equal to one and a min path with all components equal to zero for the chosen $\mathbf{x}$. Thus, there are two min paths with no common components. Contradiction. //

Structures in $\mathcal{S}_2$ have $\rho_C(\mathbf{x})=0$ for all $\mathbf{x}$. This means that $\varphi\in\mathcal{S}_2$ cannot have both $\varphi(\mathbf{x})=1$ and $\varphi(\mathbf{1}-\mathbf{x})=1$. In other words, $\varphi\in\mathcal{S}_2$ is "never contradictory" but may be "blocked" since it is possible to have $\varphi(\mathbf{x})=0$ and $\varphi(\mathbf{1}-\mathbf{x})=0$.

Proposition 1.3

$$\overline{\mathbf{S}}_2=\{\varphi(\mathbf{x})\mid\rho_B(\mathbf{x})=0,\ \text{for all }\mathbf{x}\}$$

Proof: Similar to that of Proposition 1.2 (Replace arguments of min paths for min cuts.) //

Structures in $\overline{\mathbf{S}}_2$ have $\rho_B(\mathbf{x})=0$ for all $\mathbf{x}$. This means that structure $\varphi\in\overline{\mathbf{S}}_2$ cannot have $\varphi(\mathbf{x})=0$ and $\varphi(\mathbf{1}-\mathbf{x})=0$. In words, φ is "never blocked" but may be "contradictory" since it is possible that $\varphi(\mathbf{x})=1$ and $\varphi(\mathbf{1}-\mathbf{x})=1$.

Definition (Class $\boldsymbol{M}$)

$$\boldsymbol{M} \equiv \mathcal{S}_2 \cap \overline{\mathrm{S}}_2$$

Example 1.4

A 3 out of 5 structure belongs to $\boldsymbol{M}$

An equivalent definition of $\boldsymbol{M}$ is now proven,

Proposition 1.5

$$\boldsymbol{M} = \{\varphi(\mathbf{x}) \mid \varphi(\mathbf{x}) = \varphi^D(\mathbf{x})\}$$

In words: $\boldsymbol{M}$ consists of the self dual structures.

Proof:

(i) If $\varphi(\mathbf{x}) = \varphi^D(\mathbf{x})$ then

$$\begin{aligned}\rho_C(\mathbf{x}) = \varphi(\mathbf{x})\varphi(\mathbf{1}-\mathbf{x}) &= \varphi(\mathbf{x})(1-\varphi^D(\mathbf{x})) \\ &= \varphi(\mathbf{x}) - \varphi(\mathbf{x})\ \varphi^D(\mathbf{x}) \\ &= \varphi(\mathbf{x}) - \varphi(\mathbf{x}) = 0 \text{ for all } \mathbf{x}\end{aligned}$$

$$\begin{aligned}\rho_B(\mathbf{x}) = \varphi^D(\mathbf{x})\varphi^D(\mathbf{1}-\mathbf{x}) &= \varphi(\mathbf{x})\varphi(\mathbf{1}-\mathbf{x}) \\ &= \rho_C(\mathbf{x}) = 0\end{aligned}$$

Therefore, $\varphi(\mathbf{x}) \in \mathcal{S}_2$ and $\varphi(\mathbf{x}) \in \overline{\mathrm{S}}_2$ and thus $\varphi(\mathbf{x}) \in \mathcal{S}_2 \cap \overline{\mathrm{S}}_2$

(ii) If $\rho_C(\mathbf{x}) = 0$ and $\rho_B(\mathbf{x}) = 0$ for all $\mathbf{x}$ then

$$\rho_C(\mathbf{x}) = \varphi(\mathbf{x})\varphi(\mathbf{1}-\mathbf{x}) = 0 \text{ and}$$

$$\begin{aligned}\rho_B(\mathbf{x}) &= \varphi^D(\mathbf{x})\varphi^D(\mathbf{1}-\mathbf{x}) \\ &= (1-\varphi(\mathbf{1}-\mathbf{x}))(1-\varphi(\mathbf{x}))\end{aligned}$$

$$=1+\varphi(\mathbf{x})\varphi(\mathbf{1}-\mathbf{x})-\varphi(\mathbf{x})-\varphi(\mathbf{1}-\mathbf{x})=0$$

Substituting the first into the second we obtain,

$$1-\varphi(\mathbf{x})-\varphi(\mathbf{1}-\mathbf{x})=0 \text{ or } \varphi(\mathbf{x})=\varphi^D(\mathbf{x}) \qquad //$$

Pictorially we have,

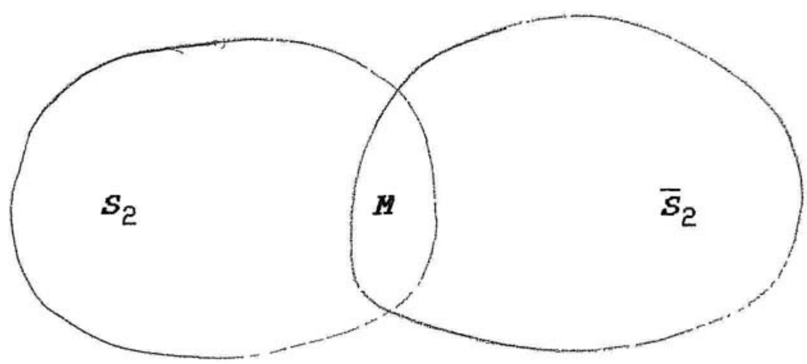

Figure 1.1: The classes $\mathcal{S}_2$, $\overline{\mathrm{S}}_2$, $\mathcal{M}$

The definitions of $\mathcal{S}_2$ and $\overline{\mathrm{S}}_2$ can be generalized as follows,

Definition (Class $\mathcal{S}_p$)

$\mathcal{S}_p$ is the class of coherent structures whose elements are the structures which have the property that any p min paths have at least one common component.

Note: If the number of min paths of the structure is m<p , then if it belongs to $\mathcal{S}_m$ we say that it also belongs to $\mathcal{S}_p$ p>m.

Definition (Class $\overline{\mathrm{S}}_p$)

$\overline{\mathrm{S}}_p$ is the class of coherent structures whose elements are

the structures which have the property that any p min cuts have at least one common component.

Note: If the number of min cuts of the structure is $m<p$, then if it belongs to $\overline{S}_m$, we say that it also belongs to $\overline{S}_p$ $p>m$.

For $p=2$ we obtain the usual definition of $\mathcal{S}_2$ and $\overline{S}_2$

Certainly, $\varphi\in\mathcal{S}_p \iff \varphi^D\in\overline{S}_p$

Also $\mathcal{S}_p\subset\mathcal{S}_{p-1}$ since, whenever any p min paths have at least one common component, it follows that any p-1 min paths will have the same component in common.

Similarly, $\overline{S}_p\subset\overline{S}_{p-1}$

Lemma 1.6

$\mathcal{S}_3\cap\overline{S}_3$ contains only the one component structure.

Proof:

Suppose that $\varphi\in\mathcal{S}_3\cap\overline{S}_3$ then since $\mathcal{S}_3\subset\mathcal{S}_2$ and $\overline{S}_3\subset\overline{S}_2$, it follows that $\mathcal{S}_3\cap\overline{S}_3\subset\boldsymbol{M}$ and thus φ must satisfy $\varphi(\mathbf{x})=\varphi^D(\mathbf{x})$ for all $\mathbf{x}$.

Pick min paths P , Q of φ which have common components $B=\{x_{j(1)},\ldots,x_{j(b)}\}$, where the notation $x_{j(i)}$ is used in this book to mean that i is a subscript of j which in turn is a subsript of x. Set B has at least one element by assumption that any 3 min paths have at least one common component. Now all other min paths of the structure must have a non empty intersection with B since any three min paths have at least one common component. It follows that B is a cut of the structure φ. Therefore, there is a subset $B'\subseteq B$ which is a min cut of

the structure. But $\varphi(\mathbf{x})=\varphi^D(\mathbf{x})$ for all $\mathbf{x}$ and therefore, B' is also a min path of φ. However, B and hence B' is a subset both of P and Q. But since B', P, Q are all min paths it cannot hold that $B' \subset P$ or $B' \subset Q$. It follows that $B'=P=Q$. But then all min paths of φ must be identical and therefore, φ, having no redundant paths, is a series structure. But the only series structure that respects $\varphi(\mathbf{x})=\varphi^D(\mathbf{x})$ is the one component structure. //

Theorem 1.7

$\mathcal{S}_p \cap \overline{\mathbf{S}}_p$ for $p \geq 3$ contains only the one component structure.

Proof:

In Lemma 1.6 it was shown that the statement holds for $p=3$. For $p>3$, we first observe that the one component structure belongs to $\mathcal{S}_p \cap \overline{\mathbf{S}}_p$ since it satisfies both the definition of $\mathcal{S}_p$ and $\overline{\mathbf{S}}_p$. We know that $\mathcal{S}_p \subset \mathcal{S}_{p-1}$ and therefore, $\mathcal{S}_p \subset \mathcal{S}_3$. The same holds for $\overline{\mathbf{S}}_p \subset \overline{\mathbf{S}}_3$. Therefore, $\mathcal{S}_p \cap \overline{\mathbf{S}}_p \subseteq \mathcal{S}_3 \cap \overline{\mathbf{S}}_3$. But $\mathcal{S}_3 \cap \overline{\mathbf{S}}_3$ contains only the one component structure by Lemma 1.6 then $\mathcal{S}_p \cap \overline{\mathbf{S}}_p$ contains at most the one component structure; and it does as we showed above. //

Pictorially we have,

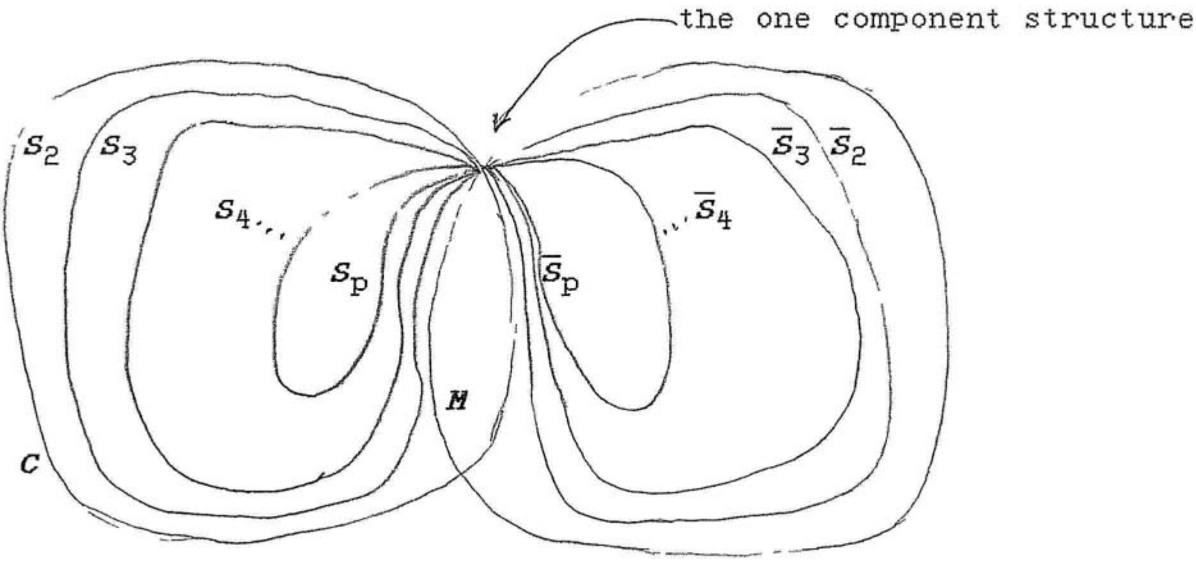

Figure 1.2: The classes S_p, $\overline{S}_p$, M

Remarks

1) It can be observed that as p tends to ∞ while n remains finite, S_p tends to become a set containing only one component cut structures. In antipode $\overline{S}_p$ tends to contain only one component path structures.

2) We can trivially define $S_1 = \overline{S}_1 = C =$ the class of all coherent structures

1.2 Definitions: Acceptance, Contradiction, Blockedness, Rejection at Structure Level

In this section the notion of "blockedness" and "contradiction" of a structure φ that were mentioned in the last section are generalized and properly defined.

Instead of a single $\mathbf{x}$ and its negation $\mathbf{1}-\mathbf{x}$, we will refer to a group of $\mathbf{x}_i$'s: $\{\mathbf{x}_i\}_{i=1}^p$ and make use of their conjunction, $\Pi^p{}_{i=1}\mathbf{x}_i$, or their disjunction, $\sqcup^p{}_{i=1}\mathbf{x}_i$ $(\equiv 1-\Pi^p{}_{i=1}(\mathbf{1}-\mathbf{x}_i))$, where the following notation has been used:

$$\mathbf{x}_1\mathbf{x}_2\equiv(x^1{}_1x^1{}_2,\ x^2{}_1x^2{}_2,\ldots,x^n{}_1x^n{}_2)$$

$$\mathbf{x}_1\sqcup\mathbf{x}_2\equiv(x^1{}_1\sqcup x^1{}_2,x^2{}_1\sqcup x^2{}_2,\ldots,x^n{}_1\sqcup x^n{}_2)$$

where

$$x_1\sqcup x_2\equiv 1-(1-x_1)(1-x_2)$$

I. Definitions using conjunction of $\mathbf{x}_1,\ldots,\mathbf{x}_p$

Definition I.1

A coherent structure φ *strongly passes* the conjunction of $\mathbf{x}_1,\ldots,\mathbf{x}_p$ iff $\varphi(\mathbf{x}_1)=1,\quad \varphi(\mathbf{x}_2)=1,\ldots,\varphi(\mathbf{x}_p)=1$ and $\varphi(\mathbf{1}-\Pi^p{}_{i=1}\mathbf{x}_i)=0$

Define now,

$$\rho_{Tp}(\varphi;\mathbf{x}_1,\ldots,\mathbf{x}_p)\equiv$$

$$\equiv\Pi^p{}_{i=1}\varphi(\mathbf{x}_i)[1-\varphi(\mathbf{1}-\Pi^p{}_{i=1}\mathbf{x}_i)]=$$

$$=\Pi^{p}_{i=1}\varphi(\mathbf{x}_i)\ \ \varphi^{D}(\Pi^{p}_{i=1}\mathbf{x}_i)$$

then it follows that Definition I.1 can also be expressed as,

φ *strongly passes* the conjunction of $\mathbf{x}_1,\ldots,\mathbf{x}_p$ iff $\rho_{Tp}(\varphi;\mathbf{x}_1,\ldots,\mathbf{x}_p)=1$

In words: φ passes each of the $\mathbf{x}_i$'s but the negation of their conjunction does not pass the structure φ.

Definition I.2

A coherent structure φ is *contradictory* for the conjunction $\mathbf{x}_1,\ldots,\mathbf{x}_p$ iff $\varphi(\mathbf{x}_1)=1,\ldots,\varphi(\mathbf{x}_p)=1$ and $\varphi(\mathbf{1}-\Pi^{p}_{i=1}\mathbf{x}_i)=1$

Define now,

$$\rho_{Cp}(\varphi;\mathbf{x}_1,\ldots,\mathbf{x}_p)\equiv$$

$$\equiv\Pi^{p}_{i=1}\varphi(\mathbf{x}_i)\,\varphi(\mathbf{1}-\Pi^{p}_{i=1}\mathbf{x}_i).$$

Then we can rephrase Definition I.2:

A coherent structure φ is *contradictory* for the conjunction of $\mathbf{x}_1,\ldots,\mathbf{x}_p$ iff $\rho_{Cp}(\varphi;\mathbf{x}_1,\ldots,\mathbf{x}_p)=1$

In words: φ passes each of the $\mathbf{x}_i$'s, but also the negation of their conjunction passes.

Definition I.3

A coherent structure φ is *blocked* for the conjunction of $\mathbf{x}_1,\ldots,\mathbf{x}_p$ iff there exists $k\in\{1,2,\ldots,p\}$ for which $\varphi(\mathbf{x}_k)=0$ and $\varphi(\mathbf{1}-\Pi^{p}_{i=1}\mathbf{x}_i)=0$ or equivalently iff $\rho_{Bp}(\varphi;\mathbf{x}_1,\ldots,\mathbf{x}_p)=1$ where,

$$\rho_{Bp}(\varphi;\mathbf{x}_1,\ldots,\mathbf{x}_p)\equiv$$

$$\equiv[1-\Pi^{p}_{i=1}\varphi(\mathbf{x}_i)]\ \ \varphi^{D}(\Pi^{p}_{i=1}\mathbf{x}_i)$$

In words: At least one of the $\mathbf{x}_i$'s is rejected by φ (does not

pass φ) and thus their conjunction, $\Pi^p_{i=1}\mathbf{x}_i$; but the negation of their conjunction does not pass either.

Definition I.4

A coherent structure φ *strongly rejects* the conjunction of $\mathbf{x}_1,\ldots,\mathbf{x}_p$ iff there exists $k\in\{1,\ldots,p\}$ for which $\varphi(\mathbf{x}_k)=0$ and $\varphi(\mathbf{1}-\Pi^p_{i=1}\mathbf{x}_i)=1$ or equivalently iff $\rho_{Fp}(\varphi;\mathbf{x}_1,\ldots,\mathbf{x}_p)=1$ where

$\rho_{Fp}(\varphi;\mathbf{x}_1,\ldots,\mathbf{x}_p)\equiv$

$$\equiv[1-\Pi^p_{i=1}\varphi(\mathbf{x}_i)]\varphi(\mathbf{1}-\Pi^p_{i=1}\mathbf{x}_i)$$

In words: the conjunction $\Pi^p_{i=1}\mathbf{x}_i$ does not pass, while the negation of the conjunction passes.

II. Definitions using the disjunction of $\mathbf{x}_1,\ldots,\mathbf{x}_p$

Definition II.1

A coherent structure φ *strongly passes* the disjunction of $\mathbf{x}_1,\ldots,\mathbf{x}_p$ iff there exists $k\in\{1,\ldots,p\}$ for which $\varphi(\mathbf{x}_k)=1$ and $\varphi(\mathbf{1}-\amalg^p_{i=1}\mathbf{x}_i)=0$ or equivalently iff $\sigma_{Tp}(\varphi;\mathbf{x}_1,\ldots,\mathbf{x}_p)=1$ where

$$\sigma_{Tp}(\varphi;\mathbf{x}_1,\ldots,\mathbf{x}_p)\equiv\amalg^p_{i=1}\varphi(\mathbf{x}_i)\,\varphi^D(\amalg^p_{i=1}\mathbf{x}_i)$$

In words: At least one of the $\mathbf{x}_i$'s passes, and thus their disjunction; but its negation does not pass.

Definition II.2

A coherent structure φ is *contradictory* for the disjunction of $\mathbf{x}_1,\ldots,\mathbf{x}_p$ iff there exists $k\in\{1,\ldots,p\}$ for which

$\varphi(\mathbf{x}_K)=1$ and $\varphi(\mathbf{1}-\sqcup^p{}_{i=1}\mathbf{x}_i)=1$ or equivalently iff $\sigma_{Cp}(\varphi;\mathbf{x}_1,\ldots,\mathbf{x}_p)=1$ where

$\sigma_{Cp}(\varphi;\mathbf{x}_1,\ldots,\mathbf{x}_p)\equiv\sqcup^p{}_{i=1}\varphi(\mathbf{x}_i)\,\varphi(\mathbf{1}-\sqcup^p{}_{i=1}\mathbf{x}_i)$

In words: both the disjunction and its negation pass .

Definition II. 3

A coherent structure φ is *blocked* for the disjunction of $\mathbf{x}_1,\ldots,\mathbf{x}_p$ iff $\varphi(\mathbf{x}_1)=0,\ldots,\varphi(\mathbf{x}_p)=0$ and $\varphi(\mathbf{1}-\sqcup^p{}_{i=1}\mathbf{x}_i)=0$ or iff $\sigma_{Bp}(\varphi;\mathbf{x}_1,\ldots,\mathbf{x}_p)=1$ where

$\sigma_{Bp}(\varphi;\mathbf{x}_1,\ldots,\mathbf{x}_p)\equiv\Pi^p{}_{i=1}\varphi^D(\mathbf{1}-\mathbf{x}_i)\,\varphi^D(\sqcup^p{}_{i=1}\mathbf{x}_i)$

In words: both the disjunction and its negation are rejected.

Definition II. 4

A coherent structure φ *strongly rejects* the disjunction of $\mathbf{x}_1,\ldots,\mathbf{x}_p$ iff $\varphi(\mathbf{x}_1)=0,\ldots,\varphi(\mathbf{x}_p)=0$ and $\varphi(\mathbf{1}-\sqcup^p{}_{i=1}\mathbf{x}_i)=1$ or iff $\sigma_{Fp}(\varphi;\mathbf{x}_1,\ldots,\mathbf{x}_p)=1$ where

$\sigma_{Fp}(\varphi;\mathbf{x}_1,\ldots,\mathbf{x}_p)\equiv[1-\sqcup^p{}_{i=1}\varphi(\mathbf{x}_i)]\,\varphi(\mathbf{1}-\sqcup^p{}_{i=1}\mathbf{x}_i)$

In words: the disjunction is rejected but its negation passes.

The functions $\rho_{Tp}(\varphi;\mathbf{x}_1,\ldots,\mathbf{x}_p)$, $\rho_{Bp}(\varphi;\mathbf{x}_1,\ldots,\mathbf{x}_p)$, $\sigma_{Fp}(\varphi;\mathbf{x}_1,\ldots,\mathbf{x}_p)$, etc. will be referred to as *indicators*.

Discussion

In definition I. 1, at first thought we might consider defining "strongly passes" the conjunction of $\mathbf{x}_1,\ldots,\mathbf{x}_p$ when $\varphi(\mathbf{x}_1)=1,\ldots,\varphi(\mathbf{x}_p)=1$ and $\varphi(\Pi^p{}_{i=1}\mathbf{x}_i)=1$. But this is too strong a requirement and leads to demanding that φ be a

series structure. A similar argument applies to definition II.4, where if we define that φ "strongly rejects" the disjunction of $\mathbf{x}_1,\dots,\mathbf{x}_p$ iff $\varphi(\mathbf{x}_1)=0,\dots,\varphi(\mathbf{x}_p)=0$ and $\varphi(\sqcup^p{}_{i=1}\mathbf{x}_i)=0$ leads to demanding that φ be a parallel structure.

For each group of $\mathbf{x}_1,\dots,\mathbf{x}_p$, one and only one of the indicators of the conjunction group is equal to 1 and again one and only one of the indicators of the disjunction group equals 1. In words this says that for any particular choice of $\mathbf{x}_1,\dots,\mathbf{x}_p$ a coherent structure φ will exclusively either "strongly pass" or "strongly reject" or be "contradictory" or be "blocked" regarding the conjunction of $\mathbf{x}_1,\dots,\mathbf{x}_p$; and will exclusively either "strongly pass" or be "contradictory" or be "blocked" regarding the disjunction of $\mathbf{x}_1,\dots,\mathbf{x}_p$.

The following relations hold among indicator structures,

$\rho_{Tp}(\varphi;\mathbf{x}_1,\dots,\mathbf{x}_p)=\sigma_{Fp}(\varphi^D;\mathbf{1}-\mathbf{x}_1,\dots,\mathbf{1}-\mathbf{x}_p)$

$\rho_{Cp}(\varphi;\mathbf{x}_1,\dots,\mathbf{x}_p)=\sigma_{Bp}(\varphi^D;\mathbf{1}-\mathbf{x}_1,\dots,\mathbf{1}-\mathbf{x}_p)$

$\rho_{Bp}(\varphi;\mathbf{x}_1,\dots,\mathbf{x}_p)=\sigma_{Cp}(\varphi^D;\mathbf{1}-\mathbf{x}_1,\dots,\mathbf{1}-\mathbf{x}_p)$

$\rho_{Fp}(\varphi;\mathbf{x}_1,\dots,\mathbf{x}_p)=\sigma_{Tp}(\varphi^D;\mathbf{1}-\mathbf{x}_1,\dots,\mathbf{1}-\mathbf{x}_p)$

These relations hold because of symmetry. As an example, a proof for one of them is presented,

$$
\begin{aligned}
&\sigma_{Bp}(\varphi;\mathbf{x}_1,\dots,\mathbf{x}_p)\equiv\\
&\equiv\Pi^p{}_{i=1}(1-\varphi(\mathbf{x}_i))\,[1-\varphi(\mathbf{1}-\sqcup^p{}_{i=1}\mathbf{x}_i)]=\\
&=\Pi^p{}_{i=1}\varphi^D(\mathbf{1}-\mathbf{x}_i)\,[\varphi^D(\mathbf{1}-\Pi^p{}_{i=1}(\mathbf{1}-\mathbf{x}_i))]=\\
&=\rho_{Cp}(\varphi^D;\mathbf{1}-\mathbf{x}_1,\dots,\mathbf{1}-\mathbf{x}_p)
\end{aligned}
$$

When $p=1$ then the ρ and σ indicators are identical as

there is no meaning to conjunction or disjunction and we are back to the case of section 1.1.

1.3 Relation of Classes $S_p, \bar{S}_p$ M to Contradiction and Blockedness

Theorem 1.8

$S_{p+1}=\{\varphi\in C \mid \rho_{Cp}(\varphi;\mathbf{x}_1,\ldots,\mathbf{x}_p)=0; \forall\ \mathbf{x}_1,\ldots,\mathbf{x}_p\}$

(In words: S_{p+1}, which was earlier defined as the set of structures whose any p+1 paths have a common component, is the set of structures which are never contradictory under conjunction of *any* p-element group of x_i's: $\mathbf{x}_1,\ldots,\mathbf{x}_p$

Proof:

(i) Let $\varphi\in S_{p+1}$. Suppose to the contrary that there is a group $\mathbf{x}_1,\ldots,\mathbf{x}_p$ so that $\rho_{Cp}=1$, then $\varphi(\mathbf{x}_1)=1,\ldots,$ $\varphi(\mathbf{x}_p)=1$ and $\varphi(\mathbf{1}-\Pi^p{}_{i=1}\mathbf{x}_i)=1$. For each $\mathbf{x}_i$ there must be a path that passes it. Thus there are at most p+1 different min paths, one for each $\mathbf{x}_i$ and one for $(\mathbf{1}-\Pi^p{}_{i=1}\mathbf{x}_i)$. But they must have at least one common component among them, say k. It follows that $x_1{}^k=1$ and $x_2{}^k=1$ and...and $x_p{}^k=1$ and $1-\Pi^p{}_{i=1}x^k{}_i=1$ which is a contradiction. Therefore, $\rho_{Cp}(\varphi;\mathbf{x}_1,\ldots,\mathbf{x}_p)=0$

(ii) Let $\rho_{Cp}(\varphi;\mathbf{x}_1,\ldots,\mathbf{x}_p)=0$ for all $\mathbf{x}_1,\ldots,\mathbf{x}_p$. Suppose $\varphi\notin S_{p+1}$. Pick any p+1 min paths of φ. At most p of them have a common component. Choose $\mathbf{x}_1,\ldots,\mathbf{x}_p$ as follows:

$\mathbf{x}_1$ has all its components equal 0 except those corresponding to P_1 (the first min path of φ).

$\mathbf{x}_2$ has all components equal 0 except those corresponding to P_2.

$\mathbf{x}_p$ has all components 0 except those corresponding to P_p

Now $\Pi^p{}_{i=1}\mathbf{x}_i$ will have all components equal 0 except those common to all p min paths. Therefore, $\mathbf{1}-\Pi^p{}_{i=1}\mathbf{x}_i$ has all components equal to one except those that are common to all p min paths. Now by assumption there is a p+1 min path which does not have the same common components with the rest of the p paths. Hence $\mathbf{1}-\Pi^p{}_{i=1}\mathbf{x}_i$ makes all the components of the p+1 path equal 1. Thus $\varphi(\mathbf{1}-\Pi^p{}_{i=1}\mathbf{x}_i)=1$. But then $\rho_{Cp}(\varphi;\mathbf{x}_1,\ldots,\mathbf{x}_p)=1$. Contradiction. Therefore, $\varphi\in\mathcal{S}_{p+1}$. //

In the proof above it was assumed that φ has at least p+1 paths. But this is not necessary. The arguments hold even if φ has m<p+1 paths because of the definition of $\mathcal{S}_{p+1}$. In this case φ will be a one component cut structure (all its min paths will have at least one common component)

Theorem 1.9

$\overline{\mathcal{S}}_{p+1}=\{\varphi\in\mathcal{C}\,|\,\sigma_{Bp}(\varphi;\mathbf{x}_1,\ldots,\mathbf{x}_p)=0,\ \forall\ \mathbf{x}_1,\ldots,\mathbf{x}_p\}$

Proof:

Observe that

$\sigma_{Bp}(\varphi;\mathbf{x}_1,\ldots,\mathbf{x}_p)=\rho_{Cp}(\varphi^D;\mathbf{1}-\mathbf{x}_1,\ldots,\mathbf{1}-\mathbf{x}_p)$

and since $\varphi\in\overline{\mathcal{S}}_{p+1}$ <=> $\varphi^D\in\mathcal{S}_{p+1}$, the theorem is proved by symmetry of arguments. //

Proposition 1.10

$\overline{\mathcal{S}}_2=\{\varphi\in\mathcal{C}\,|\,\rho_{Bp}(\varphi;\mathbf{x}_1,\ldots,\mathbf{x}_p)=0,$ for all $p\geq 1,$

for all $\mathbf{x}_1, \ldots, \mathbf{x}_p\}$

In words: $\bar{S}_2$, defined as the set of structures whose any two min cuts have at least one common component, is also the set of structures that are never blocked under conjunction of any group of $\mathbf{x}_1, \ldots, \mathbf{x}_p$ for any $p \geq 1$.

Proof:

(i) $\rho_{Bp}(\varphi; \mathbf{x}_1, \ldots, \mathbf{x}_p) = 0$ for all $\mathbf{x}_1, \ldots, \mathbf{x}_p$ and for any $p \geq 1$. Then pick $\mathbf{x}_1, \ldots, \mathbf{x}_p$ so that $\varphi(\mathbf{x}_1) = 0, \ldots, \varphi(\mathbf{x}_p) = 0$, then $\varphi(\mathbf{1} - \Pi^p{}_{i=1}\mathbf{x}_i) = 1$. However, $\varphi(\mathbf{x}_i) = 0$ for $i = 1, \ldots, p$ implies that $\varphi(\Pi^p{}_{i=1}\mathbf{x}_i) = 0$ because φ is coherent ($\Pi^p{}_{j=1}\mathbf{x}_j \leq \mathbf{x}_i$ for all i). Now let $\Pi^p{}_{i=1}\mathbf{x}_i = \mathbf{x}$ then we have $\varphi(\mathbf{x}) = 0$ and $\varphi(\mathbf{1} - \mathbf{x}) = 1$ which implies that $\rho_{B1}(\varphi; \mathbf{x}) = 1$ for all $\mathbf{x}$ and hence $\varphi \in \bar{S}_2$.

(ii) $\varphi \in \bar{S}_2$. Suppose to the contrary that $\rho_{Bp}(\varphi; \mathbf{x}_1, \ldots, \mathbf{x}_p) = 1$ for some $\mathbf{x}_1, \ldots, \mathbf{x}_p$. Then $\varphi(\mathbf{x}_1) = \ldots = \varphi(\mathbf{x}_p) = 0$ and $\varphi(\mathbf{1} - \Pi^p{}_{i=1}\mathbf{x}_i) = 0$. On the other hand, $\varphi \in \bar{S}_2 \Rightarrow \rho_{B1} = 0 \Rightarrow$
$\Rightarrow (1 - \varphi(\mathbf{1} - \mathbf{x}))(1 - \varphi(\mathbf{x})) = 0$. Letting $\Pi^p{}_{i=1}\mathbf{x}_i = \mathbf{x}$ it follows that $\varphi(\Pi^p{}_{i=1}\mathbf{x}_i) = 1$. But φ is coherent and hence $1 = \varphi(\Pi^p{}_{i=1}\mathbf{x}_i) \leq \varphi(\mathbf{x}_1) = 0$. Contradiction. Hence $\rho_{Bp}(\varphi; \mathbf{x}_1, \ldots, \mathbf{x}_p) = 0$. //

Recall that $\bar{S}_2 \supset \bar{S}_{p+1}$ for $p \geq 2$ and therefore $\bar{S}_{p+1}$ has both $\sigma_{Bp}(\varphi; \mathbf{x}_1, \ldots, \mathbf{x}_p) = \rho_{Bp}(\varphi; \mathbf{x}_1, \ldots, \mathbf{x}_p) = 0$ and it is never blocked under conjunction or disjunction at level p.

Proposition 1.11

$S_2 = \{\varphi \in C \mid \sigma_{Cp}(\varphi; \mathbf{x}_1, \ldots, \mathbf{x}_p) = 0,$ for all

$\mathbf{x}_1, \ldots, \mathbf{x}_p$, for any $p \geq 1\}$

In words: $\mathcal{S}_2$ (the set of structures for which any two min paths have at least one common component) is the set of structures which are never contradictory under disjunction of any group $\mathbf{x}_1, \ldots, \mathbf{x}_p$ for any $p \geq 1$.

Proof:

As in Proposition 1.10 because

$\rho_{Bp}(\varphi; \mathbf{x}_1, \ldots, \mathbf{x}_p) = \sigma_{Cp}(\varphi^D; \mathbf{1}-\mathbf{x}_1, \ldots, \mathbf{1}-\mathbf{x}_p)$ and

$\varphi \in \mathcal{S}_2 \iff \varphi^D \in \overline{\mathcal{S}}_2$. //

4

Chapter 2

STATEMENTS AND LOGIC

2.1 Logic Operations (a Short Review)

Let a statement be denoted by the symbol a or a_i. Let also,

$A \equiv \{a_1, \ldots, a_h\}$ be a set of statements.

$\overline{a} \equiv$ NOTa be the negation of statement a.

It is assumed that,

Hypothesis N.1: NOT(NOTa) = a

Let also,

$a_1 \wedge a_2$, also denoted as a_1ANDa_2, be the statement that results from both statements a_1, a_2 (conjunction of a_1, a_2)

$a_1 \vee a_2$, also denoted as a_1ORa_2, be the statement that results from either a_1 or a_2, or both (disjunction of a_1, a_2).

The AND operation is assumed to have the following properties:

Hypothesis A.1: a_1ANDa_1=a_1

Hypothesis A.2: a_1ANDa_2=a_2ANDa_1

Hypothesis A.3: (a_1ANDa_2)ANDa_3=a_1AND(a_2ANDa_3)

Definition A.4: aAND(NOTa) will be called the *false statement* and will be denoted by f.

We can show,

Property A.5: a_1ANDf=f, where f is the false statement.

Definition A.6: t≡NOTf, where t is called the *true statement*.

The OR operations can be defined as,

Definition AO.1: a_1ORa_2≡NOT($\overline{a}_1$AND$\overline{a}_2$)

By exchanging a_1 with $\overline{a}_1$ and a_2 with $\overline{a}_2$ while applying NOT on both sides of AO.1 we obtain,

Property AO.2: NOT($\overline{a}_1$OR$\overline{a}_2$)=a_1ANDa_2

The following properties of OR operator can be shown to hold,

Property O. 1: $a_1 OR a_1 = a_1$

Property O. 2: $a_1 OR a_2 = a_2 OR a_1$

Property O. 3: $(a_1 OR a_2) OR a_3 = a_1 OR (a_2 OR a_3)$

Property O. 4: $a_1 OR (NOT a_1) = t$

Property O. 5: $a_1 OR t = t$

Further we assume,

Hypothesis AO. 3: $a_1 OR (a_1 AND a_2) = a_1$

Hypothesis AO. 4: $a_1 AND (a_2 OR a_3) = (a_1 AND a_2) OR (a_1 AND a_3)$

Then we obtain,

Hypothesis AO. 5: $a_1 AND (a_1 OR a_2) = a_1$

Hypothesis AO. 6: $a_1 OR (a_2 AND a_3) = (a_1 OR a_2) AND (a_1 OR a_3)$

We also know that any statement composed of other statements with the use of operators NOT, AND, OR can be written as an AND statement of groups of OR statements, or as an OR statement composed of groups of AND statements. (By repeated use of properties AO. 4, AO. 6).

The Implication operator, =>, is defined as,

Definition I. 1: $(a_1 => a_2) \equiv (\overline{a}_1 OR a_2)$

Definition I. 2: (equivalence of statements)

The symbolism $a_1 <=> a_2$ is defined as $[(a_1 => a_2) AND (a_2 => a_1)]$

The following properties of => can be shown to hold,

Property I. 3: $(a_1 => a_2) <=> (\overline{a}_2 => \overline{a}_1)$

Property I. 4: $(a_1 => a_2) => [(a_1 AND a_2) <=> a_1]$

Property I. 5: $(a_1 => a_2) => [(a_1 OR a_2) <=> a_2]$

Property I. 6: $[(a_1 => a_2) AND (a_2 => a_3)] => (a_1 => a_3)$

Property I. 7: $(a => b) => (\text{there is } c \text{ so that } b \wedge c = a)$

namely $c=(a\vee\overline{b})\wedge(b\vee\overline{a})$

Property I. 8: $(a=>b)=>$(there is c so that $a\vee c=b$)

namely $c=(\overline{a}\vee\overline{b})\wedge(a\vee b)$

Finally, we define the Exclusive OR (XOR) operator as,

Definition X. 1

$$a_1 XOR a_2 \equiv (NOT(a_1\wedge a_2))\wedge(a_1\vee a_2)$$

which can be shown to be equal to ,

$$a_1 XOR a_2=(\overline{a}_1\vee\overline{a}_2)\wedge(a_1\vee a_2)=(\overline{a}_1\wedge a_2)\vee(\overline{a}_2\vee a_1)$$

but we will not expand on its properties. For the interested reader, proofs of the above mentioned properties appear in Appendix A.

2.2 Sets of Statements

Let A be a set of statements, $A=\{a_1, \ldots, a_h\}$.
Suppose that a composite statement is formed by statements in A and the operators NOT, OR, AND. If we assign to each statement i of A the true or false value, then we can conclude if the composite statement is true or false. If, however, the composite statement in question is true or false regardless of whether the particular statements in A are true or false, then the composite statement will be called *identically true* or *identically false* respectively.
For example,

$$a_1 \wedge \bar{a}_1 \text{ is identically false (i-false)}$$

while

$$a_1 \vee \bar{a}_1 \text{ is identically true (i-true)}$$

Definition

The *logic hull* of A with respect to (w.r.t.) NOT, denoted as $\mathcal{L}_{NOT}[A]$ is the set of statements such that,

$$\mathcal{L}_{NOT}[A]=\{a \mid a\in A \text{ or } \bar{a}\in A\}$$

Example 2.1

Let $A=\{a_1, a_2, a_3\}$ and let no a_i be the negation of another, then ,

$$\mathcal{L}_{NOT}[A]=\{a_1, a_2, a_3, \bar{a}_1, \bar{a}_2, \bar{a}_3\}$$

In short, $\mathcal{L}_{NOT}[A]$ includes statements in A and their negations.

Definition

The *acyclic logic hull* of A w.r.t. logic operator AND, denoted as $\mathcal{L}_{AND}[A]$, is the set whose elements are the statements in A and all composite statements formed from statements in A by repetitive use of AND operator, as long as the composite statement is not identically false.

Definition

The *acyclic logic hull* of A w.r.t. logic operator OR, denoted as $\mathcal{L}_{OR}[A]$, is the set whose elements are the statements in A and all composite statements formed from statements in A by repetitive use of OR operator, as long as the composite statement is not identically true.

Definition

The *cyclic logic hull* of A w.r.t. logic operator AND (OR) denoted as $\mathcal{N}_{AND}[A]$, ($\mathcal{N}_{OR}[A]$), is the set of composite statements that are constructed from statements in A with repetitive use of AND (OR) operator as long as the composite statements is identically false (identically true).

Note that A may contain i-true or i-false statements, which by definition will appear in $\mathcal{L}_{AND}[A]$, $\mathcal{L}_{OR}[A]$ but not in the respective cyclic logic hulls of A.

Definition

A is called *acyclic* w.r.t. AND (OR) if $\mathcal{N}_{AND}[A]=\emptyset$,

$(\mathcal{N}_{OR}[A]=\emptyset)$

A is called *cyclic* w.r.t. AND (OR) if $\mathcal{N}_{AND}[A]\neq\emptyset$,

$(\mathcal{N}_{OR}[A]\neq\emptyset)$

Observe that

$$\mathcal{L}_{AND}[\mathcal{L}_{AND}[A]]=\mathcal{L}_{AND}[A]$$

$$\mathcal{L}_{AND}[\mathcal{N}_{AND}[A]]=\mathcal{N}_{AND}[A]$$

$$\mathcal{N}_{AND}[\mathcal{L}_{AND}[A]]=\mathcal{N}_{AND}[A]$$

and similarly for OR operations.

Example 2.2

Let $A=\{a_1, a_2, a_3\}$ then,

$$\mathcal{L}_{AND}[A]=\{a_1, a_2, a_3, a_1\wedge a_2, a_1\wedge a_3, a_2\wedge a_3, a_1\wedge a_2\wedge a_3\}$$

$\mathcal{L}_{AND}\mathcal{L}_{NOT}[A]=\{a_1, a_2, a_3, \bar{a}_1, \bar{a}_2, \bar{a}_3, a_1\wedge a_2, \bar{a}_1\wedge a_2, a_1\wedge\bar{a}_2, \bar{a}_1\wedge\bar{a}_2, a_1\wedge a_3, \bar{a}_1\wedge a_3, a_1\wedge\bar{a}_3, \bar{a}_1\wedge\bar{a}_3, a_2\wedge a_3, \bar{a}_2\wedge a_3, a_2\wedge\bar{a}_3, \bar{a}_2\wedge\bar{a}_3, a_1\wedge a_2\wedge a_3, \bar{a}_1\wedge a_2\wedge a_3, \bar{a}_1\wedge\bar{a}_2\wedge a_3, \bar{a}_1\wedge\bar{a}_2\wedge\bar{a}_3, a_1\wedge\bar{a}_2\wedge a_3, a_1\wedge\bar{a}_2\wedge\bar{a}_3, a_1\wedge a_2\wedge\bar{a}_3, \bar{a}_1\wedge a_2\wedge\bar{a}_3\}$

Example 2.3

Let $A=\{a_1\wedge a_2, a_1\wedge a_3, a_2\wedge a_3\}$ then,

$\mathcal{L}_{OR}[A]=\{a_1\wedge a_2, a_1\wedge a_3, a_2\wedge a_3, (a_1\wedge a_2)\vee(a_1\wedge a_3), (a_1\wedge a_2)\vee(a_2\wedge a_3), (a_1\wedge a_3)\vee(a_2\wedge a_3), (a_1\wedge a_2)\vee(a_1\wedge a_3)\vee(a_2\wedge a_3)\}$

$$\mathcal{L}_{AND}[A]=\{a_1\wedge a_2, a_1\wedge a_3, a_2\wedge a_3, a_1\wedge a_2\wedge a_3\}$$

Example 2.4

Let $A=\{a_1, a_2, a_2\wedge a_3, \mathrm{NOT}(a_1\wedge a_2)\}$ then A is cyclic w.r.t. AND because we can construct an i-false statement,

$(a_1\wedge a_2)\wedge\mathrm{NOT}(a_1\wedge a_2)=f$ and hence $\mathcal{N}_{AND}[A]\neq\emptyset$

Also,

$$\mathcal{L}_{AND}[A]=\{a_1, a_2, a_3\wedge a_2,\ NOT(a_1\wedge a_2),\ a_1\wedge a_2,\ a_1\wedge a_2\wedge a_3,$$
$$a_1\wedge NOT(a_1\wedge a_2),\ a_2\wedge NOT(a_1\wedge a_2),\ (a_2\wedge a_3)\wedge NOT(a_1\wedge a_2)\}$$

Now because of De Moivre's law (Definition AO.1 and Property AO.2) and distributive properties (AO.3 to AO.6), any statement constructed from the statements of a set A with repetitive use of AND, OR, NOT operators can be written as either the OR composition of groups of AND compositions of statements from $\mathcal{L}_{NOT}[A]$ or as the AND composition of groups of OR compositions of statements from $\mathcal{L}_{NOT}[A]$

It follows therefore that,

$$\mathcal{L}_{OR}[\mathcal{L}_{AND}[A]]=\mathcal{L}_{AND}[\mathcal{L}_{OR}[A]]$$

Also define,

$$A_H\equiv\mathcal{L}_{OR}\mathcal{L}_{AND}\mathcal{L}_{NOT}[A]$$

where A_H represents the set containing all possible statements that can be constructed from elements of the set A with repetitive use of the operators AND, OR, NOT.

Further, because of the definition of OR operator (AO.1) we conclude that,

$$\begin{aligned}\mathcal{L}_{NOT}\mathcal{L}_{AND}\mathcal{L}_{NOT}[A]&=\mathcal{L}_{NOT}\mathcal{L}_{OR}\mathcal{L}_{NOT}[A]\\&=\mathcal{L}_{AND}\mathcal{L}_{OR}\mathcal{L}_{NOT}[A]\\&=A_H\end{aligned}$$

Example 2.5

Let $A=\{a_1, a_2\}$ then

$$\mathcal{L}_{OR}\ \mathcal{L}_{AND}[A]=\{a_1, a_2,\ a_1\wedge a_2,\ a_1\vee a_2\}$$

Now let $A=\{a_1,\ a_2,\ a_3\}$ then ,

$\mathcal{L}_{OR}[\mathcal{L}_{AND}[A]] = \{a_1, a_2, a_3, a_1 \wedge a_2, a_1 \wedge a_3, a_2 \wedge a_3, a_1 \wedge a_2 \wedge a_3, a_1 \vee a_2, a_1 \vee a_3, a_2 \vee a_3, a_1 \vee a_2 \vee a_3, (a_1 \wedge a_2) \vee (a_1 \wedge a_3), (a_1 \wedge a_2) \vee (a_2 \wedge a_3), (a_1 \wedge a_3) \vee (a_2 \wedge a_3), (a_1 \wedge a_2) \vee (a_1 \wedge a_3) \vee (a_2 \wedge a_3), a_1 \vee (a_2 \wedge a_3), a_2 \vee (a_1 \wedge a_3), a_3 \vee (a_1 \wedge a_2)\}$

Definition

Given a set of statements A, we call the *complexity of a statement* $b \in \mathcal{L}_{\#}[A]$ w.r.t. # operator and set A denoted as $C_{\#}(b|A)$, the minimum number of elements of A needed to construct the statement by repetitive use of # operations, where # is any one of the operators AND, OR, NOT.

Observe that complexity is defined only for statements in $\mathcal{L}_{\#}[A]$ (not for those in $\mathcal{N}_{\#}[A]$).

Definition

Given a set of statements A, the (implied) *complexity of set A* w.r.t. # operator, denoted as $C_{\#}(A)$ is defined as the complexity of the most complex statement of $\mathcal{L}_{\#}[A]$ w.r.t. A :

$$C_{\#}(A) \equiv \max \{C_{\#}(b|A)\} \text{ where,}$$

the max is taken over all $b \in \mathcal{L}_{\#}[A]$

It follows that for any set of statements A, $(A \neq \emptyset)$, $C_{NOT}(A) = 1$ since $C_{NOT}(b|A) = 1$ for any $b \in \mathcal{L}_{NOT}[A]$

Also for any set of statements A,

$$C_{\#}(\mathcal{L}_{\#}[A]) = 1$$

Example 2.6

If $A \supseteq B$ then $C_{\#}(b|A) \leq C_{\#}(b|B)$ where $b \in \mathcal{L}_{\#}[A] \supseteq \mathcal{L}_{\#}[B]$. This is true because A contains all of b. It follows that b can be constructed by the same number of elements as from B and perhaps less.

Example 2.7

Let $A = \{a_1, a_2, a_3\}$ and let a_i's be such that none is implied by a logic composition (using AND, OR, NOT) of the other two, then,

(a) $C_{AND}(A) = 3$. This holds because the most complex statement of $\mathcal{L}_{AND}[A]$ is $a_1 \wedge a_2 \wedge a_3$.

(b) $C_{AND}(\mathcal{L}_{NOT}[A]) = 3$, because the most complex statement of $\mathcal{L}_{NOT}[A]$ is either $a_1 \wedge a_2 \wedge a_3$, or $\bar{a}_1 \wedge a_2 \wedge a_3$, or etc. (see Example 2.2)

(c) $C_{OR}(\mathcal{L}_{AND}[A]) = 3$, because the most complex statement is either $a_1 \vee a_2 \vee a_3$ or $(a_1 \wedge a_2) \vee (a_1 \wedge a_3) \vee (a_2 \wedge a_3)$ (recall Example 2.3).

Example 2.8

Let $A = \{a_1 \wedge a_2,\ a_1 \wedge a_3,\ a_2 \wedge a_3\}$ and let a_1, a_2, a_3, be such that none is a logic composite of the other two, then,

$C_{OR}(A) = 3$ while $C_{AND}(A) = 2$.

Chapter 3

PROPERTIES OF COMPONENTS

3.1 Types of Logic Consistencies (Logic Behaviour)

Each component i (or individual) expresses his wishes, when presented with a set of alternatives (statements) $A=\{a_1,\ldots,a_h\}$ through a characteristic function $x_i(.|A)$ which is defined below,

$$x_i(a|A) = \begin{cases} 1 & \text{if component i passes or accepts statement a as true when presented with a set of statements } A\text{, where } a\in A \\ 0 & \text{if component i does not pass or rejects or considers statement a to be false when presented with a set of statements } A \text{ where } a\in A \end{cases}$$

Also we assume that $x_i(t|A)=1$ and $x_i(f|A)=0$ for any set A, where t is the i-true statement and f is the i-false statement. This assumption on component behaviour establishes a common language among components. There must be common agreement in regarding as true statements those that are identically true (like $a \vee \bar{a}$) and regarding false those that are i-false (like $a \wedge \bar{a}$).

In order to proceed further more assumptions about the logic behaviour of components are needed. First some definitions will be presented and then their implications and interrelations will be

examined. Although the word "component" is used in what follows, it may as well be interchanged with any "binary valued function" i.e. a function taking the values 1 or 0. Note that not only the components but also the structure is a binary valued function. Also the word "component" is often used to mean the characteristic function x_i of the component i; but there is no danger of confusion.

Definition

A component $x_i(.\,|A)$ is called,

(a) *R-AND consistent* (Right AND consistent) iff given A,

$$x_i(a_1|A)\,x_i(a_2|A) \geq x_i(a_1 \wedge a_2|A)$$

whenever a_1, a_2, $a_1 \wedge a_2$ all belong to A.

(b) *L-AND consistent* (Left AND consistent) iff given A,

$$x_i(a_1|A)\,x_i(a_2|A) \leq x_i(a_1 \wedge a_2|A)$$

whenever a_1, a_2, $a_1 \wedge a_2$ all belong to A.

(c) *AND consistent* iff given A it is both R-AND and L-AND consistent:

$$x_i(a_1|A)\,x_i(a_2|A) = x_i(a_1 \wedge a_2|A)$$

whenever a_1, a_2, $a_1 \wedge a_2$ all belong to A.

(d) *R-OR consistent* (Right OR consistent) iff given A,

$$x_i(a_1|A) \sqcup x_i(a_2|A) \geq x_i(a_1 \vee a_2|A)$$

whenever a_1, a_2, $a_1 \vee a_2$ all belong to A.

(e) *L-OR consistent* (Left OR consistent) iff given A,

$$x_i(a_1|A) \sqcup x_i(a_2|A) \leq x_i(a_1 \vee a_2|A)$$

whenever a_1, a_2, $a_1 \vee a_2$ all belong to A.

(f) *OR consistent* iff given A it is both R-OR and L-OR consistent:

$$x_i(a_1|A) \sqcup x_i(a_2|A) = x_i(a_1 \vee a_2|A)$$

whenever a_1, a_2, $a_1 \vee a_2$ all belong to A.

In the following we drop A from $x_i(a|A)$ and we write simply $x_i(a)$ when there is no danger of confusion.

Lemma 3.1

R-AND consistency is equivalent to,

$$x_i(a_1) \geq x_i(a_1 \wedge a_2)$$

Proof:

(i) $x_i(a_1) \geq x_i(a_1)x_i(a_2)$ because x_i is binary. Further by R-AND consistency $x_i(a_1)x_i(a_2) \geq x_i(a_1 \wedge a_2)$.

(ii) $x_i(a_1) \geq x_i(a_1 \wedge a_2)$ and $x_i(a_2) \geq x_i(a_1 \wedge a_2)$ imply that $x_i(a_1)x_i(a_2) \geq x_i(a_1 \wedge a_2)$ hence R-AND consistency. //

Lemma 3.2

L-OR consistency is equivalent to,

$$x_i(a_1) \leq x_i(a_1 \vee a_2)$$

Proof: as in Lemma 3.1 . //

Note that L-AND is not equivalent to $x_i(a_1) \leq x_i(a_1 \wedge a_2)$ and also that R-OR is not equivalent to $x_i(a_1) \geq x_i(a_1 \vee a_2)$.

Proposition 3.3

R-AND is equivalent to L-OR

Proof:

(i) R-AND=>L-OR:

$x(a_1) \geq x(a_1 \wedge a_2)$. Let $a_1 = c \vee \overline{a}_2$ then

$x(c \vee \bar{a}_2) \geq x(c \vee \bar{a}_2 \wedge a_2) = x(c)$ and L-OR is proved because of Lemma 3.2

(ii) L-OR=>R-AND:

$x(a_1) \leq x(a_1 \vee a_2)$. Let $a_1 = c \wedge \bar{a}_2$ then

$x(c \wedge \bar{a}_2) \leq x(c \wedge \bar{a}_2 \vee a_2) = x(c)$ and R-AND is proved because of Lemma 3.1 //

Definition (Implication consistency (I-consistency))

A component x_i is called *Implication consistent* iff

$(a => b) => (x_i(a|A) \leq x_i(b|A))$

whenever $a, b \in A$.

Proposition 3.4

(I consistency)<=>(R-AND)<=>(L-OR)

Proof:

That R-AND<=>L-OR was shown in Proposition 3.3. It remains to prove that I<=>L-OR

(i) I=>L-OR:

We know that $a => (a \vee b)$ and by I consistency $x(a) \leq x(a \vee b)$ which is equivalent to L-OR by Lemma 3.2

(ii) L-OR=>I consistency:

let $a => b$ then there is a c so that $a = b \wedge c$ (Property I.7 of implication operator). But by R-AND $x(b \wedge c) \leq x(b)$ and hence $x(a) \leq x(b)$ which proves I consistency //

Definition

A component i is *L-NOT consistent* (Left NOT consistent) iff given A,

$x_i(a|A)=1 \Rightarrow x_i(\bar{a}|A)=0$

whenever a, $\bar{a}$ both belong to A.

Observe that equivalently we can write

$x_i(a|A)\,x_i(\bar{a}|A)=0$ or

$x_i(a|A) \leq 1-x_i(\bar{a}|A)$ which also justifies the name "left" because of the way the inequality points.

L-NOT consistency says that component i cannot both pass a statement and its negation. In this respect, he is never "contradictory". However, it is possible that $x_i(a)=0$ and $x_i(\bar{a})=0$ in which case he rejects both the statement a and its negation $\bar{a}$. In this sense it is possible that he is "blocked"

In short L-NOT consistency implies that the component is never contradictory but possibly blocked.

Definition

A component i is *R-NOT consistent* (Right NOT consistent) iff given A,

$x_i(a|A)=0 \Rightarrow x_i(\bar{a}|A)=1$

whenever a and $\bar{a}$ both belong to A.

Equivalently we can require that,

$x_i(a|A) \sqcup x_i(\bar{a}|A)=1$ or

$x_i(a|A) \geq 1-x_i(\bar{a}|A)$

R-NOT consistency says that component i is allowed to have both $x_i(a)=1$ and $x_i(a)=1$. In this sense it is never "blocked" but possibly "contradictory".

Definition

A component i is *NOT consistent* iff he is both L-NOT and

R-NOT consistent.

It follows that the necessary and sufficient condition for NOT consistency given A is,

$$x_i(\bar{a}|A) = 1 - x_i(a|A)$$

whenever a, $\bar{a}$ both belong to A..

NOT consistency says that the component is never contradictory and never blocked.

Definition

A component i is *L-NAND consistent* (Left NAND consistent) iff given A,

$$x_i(a_1|A)\,x_i(a_2|A) = 1 \Rightarrow x_i(\mathrm{NOT}(a_1 \wedge a_2)|A) = 0$$

whenever a_1, a_2, $\mathrm{NOT}(a_1 \wedge a_2)$ all belong to A.

Observe that an equivalent condition is,

$$x_i(a_1|A)\,x_i(a_2|A) \leq 1 - x_i(\mathrm{NOT}(a_1 \wedge a_2)|A)$$

A component that is L-NAND consistent will reject the negation of the conjunction of two statements, $\mathrm{NOT}(a_1 \wedge a_2)$, whenever he regards each one individually as true. Also note that L-NAND consistency implies L-NOT consistency by letting $a_2 = t$.

Definition

A component i is *R-NAND consistent* (Right NAND consistent) iff given A,

$$x_i(a_1|A)\,x_i(a_2|A) = 0 \Rightarrow x_i(\mathrm{NOT}(a_1 \wedge a_2)|A) = 1$$

whenever a_1, a_2, $\mathrm{NOT}(a_1 \wedge a_2)$ all belong to A.

An equivalent condition is,

$$x_i(a_1|A)\,x_i(a_2|A) \geq 1 - x_i(\mathrm{NOT}(a_1 \wedge a_2)|A)$$

A component that is R-NAND will pass the negation of the conjunc-

tion of two statements, $NOT(a_1 \wedge a_2)$, whenever he rejects either one or both when considered individually. By letting $a_2 = t$, we observe that R-NAND consistency implies R-NOT consistency.

Definition

A component is *NAND consistent* iff it is both L-NAND and R-NAND consistent.

It follows that x_i is NAND consistent iff given A,

$$x_i(a_1|A)\,x_i(a_2|A) = 1 - x_i(\overline{a_1 \wedge a_2}|A)$$

whenever a_1, a_2, $\overline{a_1 \wedge a_2}$ all belong to A.

Observe also that NAND consistency implies NOT consistency by letting $a_2 = t$.

Definition

A component i is *L-NOR consistent* (Left NOR consistent) iff given A,

$$x_i(a_1|A) \sqcup x_i(a_2|A) = 1 \;\Rightarrow\; x_i(NOT(a_1 \vee a_2)|A) = 0$$

whenever a_1, a_2, $NOT(a_1 \vee a_2)$ all belong to A.

Equivalently we may require that,

$$x_i(a_1|A) \sqcup x_i(a_2|A) \leq 1 - x_i(NOT(a_1 \vee a_2)|A)$$

A L-NOR consistent component will reject the negation of the disjunction of two statements, $NOT(a_1 \vee a_2)$, whenever he regards as true either or both when considered individually. Letting $a_2 = f$, it follows that L-NOR implies L-NOT consistency.

Definition

A component i is *R-NOR consistent* (Right NOR consistent) iff given A,

$$x_i(a_1|A) \sqcup x_i(a_2|A) = 0 \Rightarrow x_i(\mathrm{NOT}(a_1 \vee a_2)|A) = 1$$

whenever a_1, a_2, $\mathrm{NOT}(a_1 \vee a_2)$ all belong to A.

Equivalently we may require that,

$$x_i(a_1|A) \sqcup x_i(a_2|A) \geq 1 - x_i(\mathrm{NOT}(a_1 \vee a_2)|A)$$

A R-NOR consistent component will consider as true the negation of the disjunction of two statements, $\mathrm{NOT}(a_1 \vee a_2)$, whenever he rejects both when considered individually. Also R-NOR implies R-NOT letting $a_2 = f$.

Definition

A component i is *NOR consistent* iff he is both L-NOR and R-NOR consistent.

Therefore, NOR consistency is equivalent to,

$$x_i(a_1|A) \sqcup x_i(a_2|A) = 1 - x_i(\mathrm{NOT}(a_1 \vee a_2)|A)$$

whenever a_1, a_2, $\mathrm{NOT}(a_1 \vee a_2)$ all belong to A.

Also NOR implies NOT consistency letting $a_2 = f$.

3.2 Properties and Interrelations of Types of Logic Consistencies

Proposition 3.5

The following statements hold:

(a) (AND, NOT) <=> (NAND) <=> (NOR) <=> (OR, NOT)

(b) (R-AND) <=> (L-OR) <=> (I)

(c) (L-AND, NOT) <=> (R-OR, NOT)

(d) (R-NAND, L-NOT) <=> (L-NOR, R-NOT)

(e) (L-NAND, R-NOT) <=> (R-NOR, L-NOT)

(f) (R-AND, R-NOT) =>R-NAND

(g) (R-NAND, L-NOT) =>R-AND

(h) (L-AND, L-NOT) =>L-NAND

(i) (L-NAND, R-NOT) =>L-AND

(j) (R-OR, R-NOT) =>R-NOR

(k) (R-NOR, L-NOT) =>R-OR

(l) (L-OR, L-NOT) =>L-NOR

(m) (L-NOR, R-NOT) =>L-OR

Proof:

The proofs are straightforward applications of the definitions. We only present a few of them,

(a)

(a.1) [AND, NOT]=>NAND:

$$x(a_1)x(a_2)=x(a_1\wedge a_2)=1-x(\mathrm{NOT}(a_1\wedge a_2))$$

(a.2) NAND=>[OR, NOT]

We know that NAND =>NOT. Now,

$$x(\bar{a}_1)x(\bar{a}_2)=1-x(\mathrm{NOT}(\bar{a}_1\wedge\bar{a}_2))=1-x(a_1\vee a_2)$$

$(1-x(a_1))(1-x(a_2))=1-x(a_1 \vee a_2)$

$1-x(a_1) \sqcup x(a_2)=1-x(a_1 \vee a_2)$

$x(a_1) \sqcup x(a_2)=x(a_1 \vee x_2)$

(a. 3) [OR, NOT]=>NOR: as in (a. 1)

(a. 4) NOR=>AND: as in (a. 2)

(b) Already shown in Proposition 3. 4

(c) From definition of L-AND,

$x(\bar{a}_1)x(\bar{a}_2) \le x(\bar{a}_1 \wedge \bar{a}_2)=x(\mathrm{NOT}(a_1 \vee a_2))$

because of NOT consistency it becomes,

$1-x(a_1) \sqcup x(a_2) \le x(\mathrm{NOT}(a_1 \vee a_2))$

$x(a_1) \sqcup x(a_2) \ge 1-x(\mathrm{NOT}(a_1 \vee a_2))$ which is R-NOR

because of NOT as applied to the RHS we obtain,

$x(a_1) \sqcup x(a_2) \ge x(a_1 \vee a_2)$ which is R-OR

Similarly for the opposite.

(d) First observe that R-NAND=>R-NOT and (R-NOT, L-NOT)=>NOT then, [R-NAND, L-NOT]<=>[R-NAND, NOT]<=>[R-AND, NOT]<=>[L-OR, NOT] <=>L-NOR, NOT]<=>[L-NOR, R-NOT]

(f) R-AND requires that $x(a_1)x(a_2) \ge x(a_1 \wedge a_2)$

Case 1: $x(a_1 \wedge a_2)=1$ then $x(a_1)x(a_2)=1$ thus

$x(a_1)x(a_2) \ge 1-x(\mathrm{NOT}(a_1 \wedge a_2))$

Case 2: $x(a_1 \wedge a_2)=0$. Because of R-NOT $1-x(\mathrm{NOT}(a_1 \wedge a_2))=0$

therefore, $x(a_1)x(a_2) \ge 1-x(\mathrm{NOT}(a_1 \wedge a_2))$ //

These interrelations are summarized in the figure below:

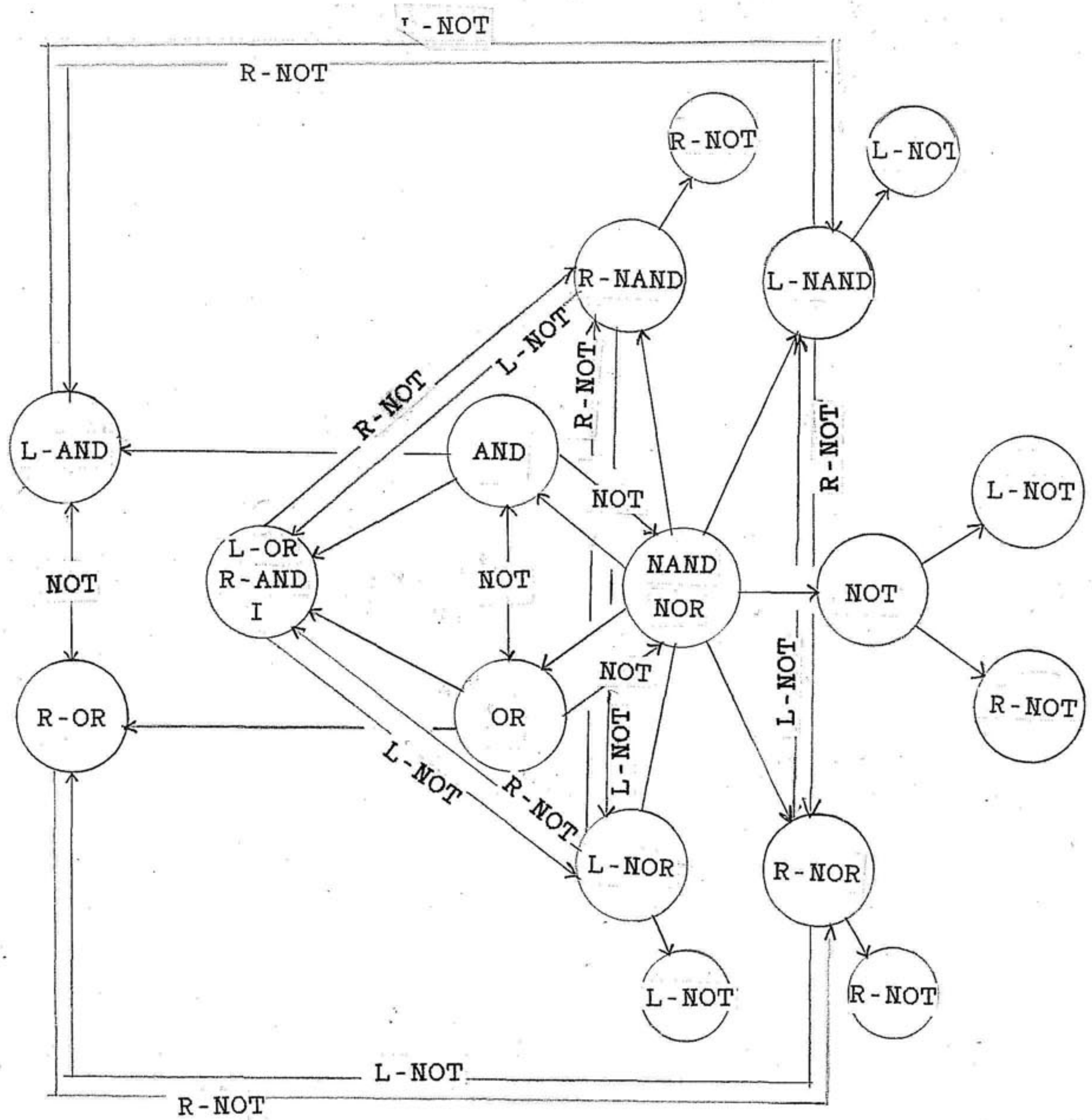

Figure 3.1: Interrelations among types of logic behaviour of components.

3.3 The Outcome Sets

When presented with a statement from a set of statements $\mathcal{A}$, component i will accept it as true or reject it as false. We can define therefore:

The *passing set* of component i when presented with the set of statements (alternatives) $\mathcal{A}$ as:

$$\mathbb{A}_i(\mathcal{A}) = \{a \in \mathcal{A} \mid x_i(a \mid \mathcal{A}) = 1\}$$

The *rejection set* of component i when presented with the set of statements $\mathcal{A}$ as,

$$\overline{\mathbb{A}}_i(\mathcal{A}) = \{a \in \mathcal{A} \mid x_i(a \mid \mathcal{A}) = 0\}$$

Certainly, $\mathbb{A}_i \cap \overline{\mathbb{A}}_i = \emptyset$ and $\mathbb{A}_i \cup \overline{\mathbb{A}}_i = \mathcal{A}$.

Suppose that $\mathcal{A}$ includes the negation of some of its statements. For example, let a and $\overline{a}$ both belong to $\mathcal{A}$. It is possible that component i may pass both a and $\overline{a}$. Or it may reject a and $\overline{a}$ etc.. We are led, therefore, to define the following sets,

the *strongly passing* or *Truth* set:

$$T_i(\mathcal{A}) = \{a \in \mathcal{A} \mid x_i(a \mid \mathcal{A}) = 1,\ x_i(\overline{a} \mid \mathcal{A}) = 0\}$$

the *Contradictory* set:

$$C_i(\mathcal{A}) = \{a \in \mathcal{A} \mid x_i(a \mid \mathcal{A}) = 1,\ x_i(\overline{a} \mid \mathcal{A}) = 1\}$$

the *Blocked* set:

$$B_i(\mathcal{A}) = \{a \in \mathcal{A} \mid x_i(a \mid \mathcal{A}) = 0,\ x_i(\overline{a} \mid \mathcal{A}) = 0\}$$

the *strong rejection* or *False* set:

$$F_i(\mathcal{A}) = \{a \in \mathcal{A} \mid x_i(a \mid \mathcal{A}) = 0,\ x_i(\overline{a} \mid \mathcal{A}) = 1\}$$

It is easy to check that these four sets are mutually exclusive. To simplify notation we will write simply T_i instead of $T_i(\mathcal{A})$

as well as for C_i, B_i, F_i.

Now,

$$\mathcal{A}_i \supseteq T_i \cup C_i \tag{3.1}$$

$$\overline{\mathcal{A}}_i \supseteq B_i \cup F_i \tag{3.2}$$

To see why (3.1) holds take a statement $b \in \mathcal{A}_i$ with $b \in A$ but $\overline{b} \notin A$ then b does not belong neither to T_i nor to C_i. Again if we pick a statement $b \in T_i \cup C_i$ then necessarily $x_i(b) = 1$ and hence $b \in \mathcal{A}_i$. For similar reasons (3.2) also holds. In case A is such that $A = \mathcal{L}_{NOT}[A]$ which implies that for each $b \in A \Rightarrow \overline{b} \in A$, then,

$$\mathcal{A}_i = T_i \cup C_i \tag{3.3}$$

and

$$\overline{\mathcal{A}}_i = B_i \cup F_i \tag{3.4}$$

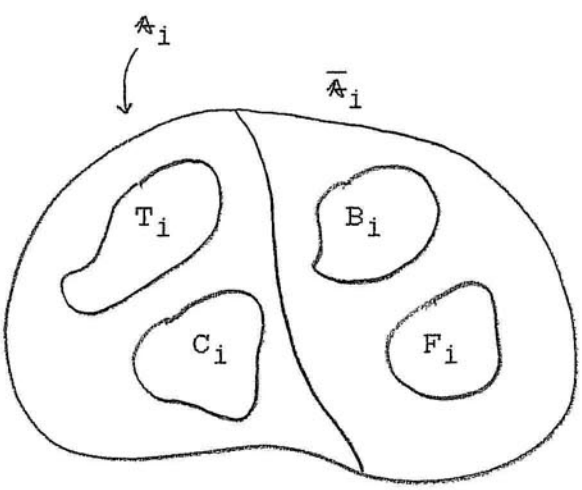

Fig. 3.2: The outcome sets (the general case)

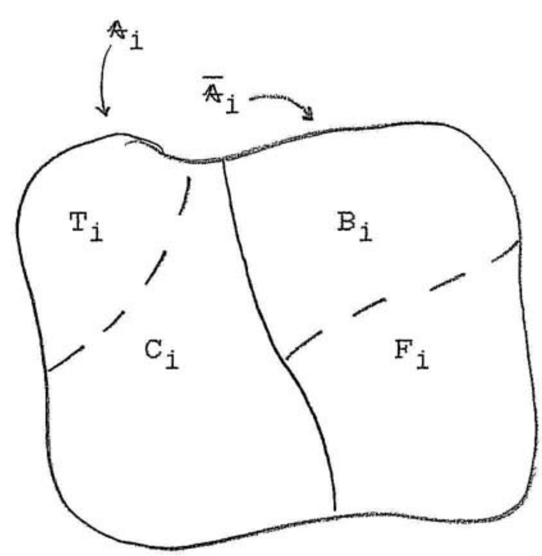

Fig. 3.3: The outcome sets when $A = \mathcal{L}_{NOT}[A]$

If component i is L-NOT consistent then we see that $C_i=\emptyset$ for any A since whenever $x_i(a)=1$, $x_i(\overline{a})=0$ by L-NOT consistency. For similar reasons $B_i=\emptyset$ if component i is R-NOT consistent. Finally, if it is NOT consistent then $B_i=C_i=\emptyset$ for any A.

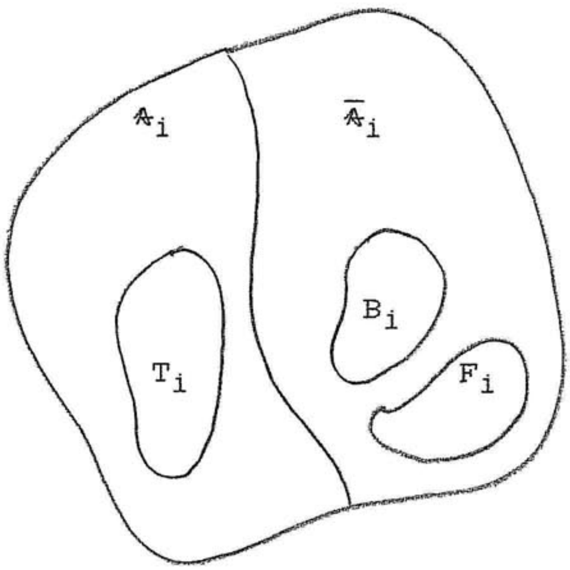

Fig. 3.4: L-NOT implies $C_i=\emptyset$

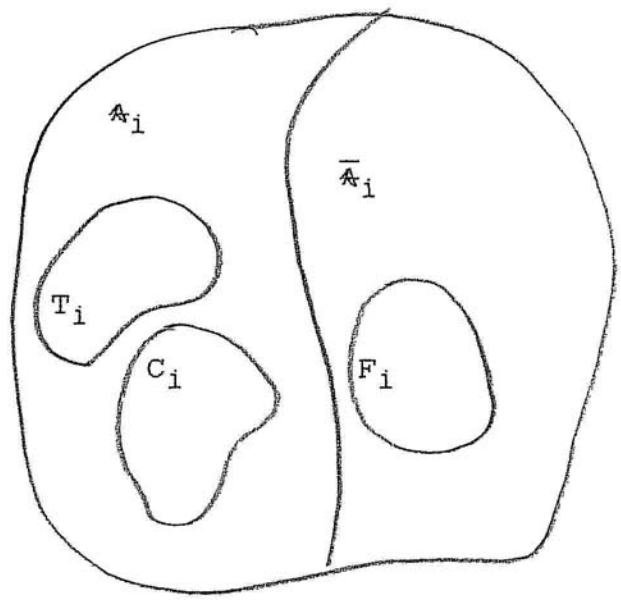

Fig. 3.5: R-NOT implies $B_i=\emptyset$

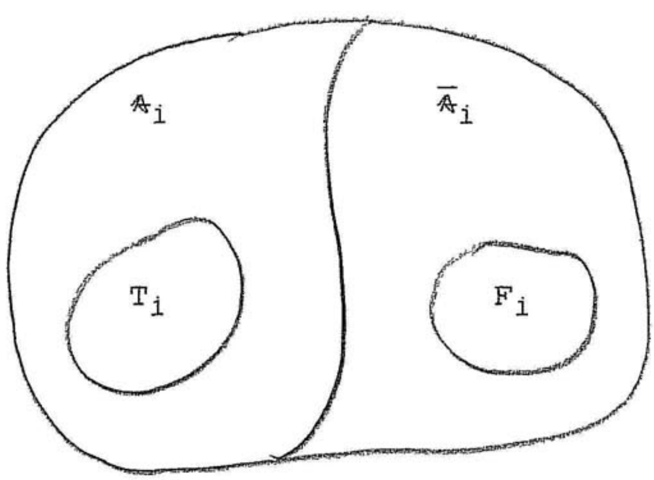

Fig. 3.6: NOT consistency implies

$C_i=B_i=\emptyset$

Note: When component i is L-NOT one is tempted to say that $\mathcal{A}_i=T_i$ because we know that for each $a\in\mathcal{A}_i$ its negation $\bar{a}$ (even if it is not in *A*) would be rejected by component i because of L-NOT consistency. However, this is not always correct, for if *A* is augmented to *A'* to include all the negations of the statements it contains, then perhaps component i will change behaviour and for some a_j for which $x_i(a_j|A)=1$ maybe now we will have $x_i(a_j|A')=0$. Therefore, only if component i is "absolute" in his decisions i.e. he decides for each statement separately regardless of *A* (his characteristic function x_i is not a function of *A*), then the augmentation of *A* to *A'* will not affect his views on what is true or false what is to be accepted and what is to be rejected. In this case, if he is also L-NOT consistent then $\mathcal{A}_i(A)=T_i(A)$ (all passing statements are strongly passing). Further, if the component is R-NOT and "absolute" then $\bar{\mathcal{A}}_i=F_i$ (all rejected statements are strongly rejected), while if he is "absolute"and NOT consistent then $\mathcal{A}_i=T_i$ and $\bar{\mathcal{A}}_i=F_i$. (All passing statements are strongly passing and all rejected are strongly rejected).
The outcome sets $\mathcal{A}_i$, $\bar{\mathcal{A}}_i$ and T_i, C_i, B_i, F_i categorize single statements of the set *A*. In the following, these ideas are generalized to groups of statements from a set *A* (their AND conjunction or OR disjunction) and appropriate outcome sets are defined for component i.

The *p-AND-passing* set:

$$\hat{\mathcal{A}}_{ip}(A)\equiv\{(a_{j(1)},\dots,a_{j(p)})\,|\,\Pi^{p}_{K=1}x_i(a_{j(K)}|A)=1,$$
$$C_{AND}(a_{j(1)}\wedge\dots\wedge a_{j(p)}\,|\,\mathcal{A}_i(A))=p\}$$

where the notation $a_{j(i)}$ is used to mean that i is a subscript of j which in turn is a subscript of a.

$\hat{\mathcal{A}}_{ip}(A)$ contains all groups of p statements that belong to $\mathcal{A}_i(A)$ (because of $\Pi^p_{K=1}x_i(a_{j(k)}|A)=1$) and whose AND complexity w.r.t. $\mathcal{A}_i$ is p. The requirement that $C(a_{j(1)}\wedge\ldots\wedge a_{j(p)}|\mathcal{A}_i)=p$ implicitly assumes that $a_{j(1)}\wedge\ldots\wedge a_{j(p)}\neq f$ (for $p\geq 2$) since complexity is not defined but for elements in $\mathcal{L}_{AND}[\mathcal{A}_i(A)]$ which by definition does not contain identically false statements except for those that are present in A (for those $p=1$).

Not all groups of p statements belong to $\hat{\mathcal{A}}_{ip}(A)$, but only those whose AND complexity is p. For example, let $a_1\in\mathcal{A}_i$, $a_2\in\mathcal{A}_i$ and $(a_1\wedge a_2)\in\mathcal{A}_i$. Now the group $(a_1, a_2, a_{j(1)},\ldots, a_{j(p-2)})$ of p statements has AND complexity w.r.t. $\mathcal{A}_i$ equal to p-1 and thus it will not appear in any of the sets $\hat{\mathcal{A}}_{iq}$, for all q. Instead, the group $((a_1\wedge a_2), a_{j(1)},\ldots, a_{j(p-2)})$ of p-1 statements has AND complexity w.r.t. $\mathcal{A}_i$ equal to p-1 and thus it will appear in $\hat{\mathcal{A}}_{ip-1}$.

It follows, therefore, that by using the notion of complexity of a statement we do not lose any of the possible conjunctions from those in $\mathcal{L}_{AND}[\mathcal{A}_i(A)]$, but consider only those groups that form such conjunctions with the least number of statements from those in $\mathcal{A}_i(A)$.

The *p-AND-rejection* set:

$\hat{\hat{\mathcal{A}}}_{ip}\equiv\{(a_{j(1)},\ldots,a_{j(p)})|\Pi^p_{K=1}x_i(a_{j(k)}|A)=0\}$

It contains the groups of p statements of A so that not all statements in each group belongs to $\mathcal{A}_i$ (i.e. at least one

statement in each group is rejected by component i). In the definition of the p-AND-rejection set we do not introduce the notion of complexity. It is not necessary as will be seen later when we discuss outcome sets at the structure level.

Similarly we define,

The *p-OR-passing*:

$\check{\mathcal{A}}_{ip}(A) \equiv \{(a_{j(1)}, \ldots, a_{j(p)}) \mid \sqcup^{p}_{K=1} x_i(a_{j(K)} \mid A) = 1\}$

It contains the group of p statements of A so that at least one belongs to $\mathcal{A}_i(A)$ (i.e. at least one statement in each group is accepted by component i).

The *p-OR-rejection* set:

$\check{\bar{\mathcal{A}}}_{ip}(A) \equiv \{(a_{j(1)}, \ldots, a_{j(p)}) \mid \sqcup^{p}_{K=1} x_i(a_{j(K)} \mid A) = 0,$

$C_{OR}(a_{j(1)} \vee \ldots \vee a_{j(p)} \mid \bar{\mathcal{A}}_i) = p\}$

It contains all groups of p statements so that each one of the statements is rejected by component i and so that the OR disjunction within each group has OR complexity w.r.t. $\bar{\mathcal{A}}_i(A)$ equal to p.

Certainly, for p=1 the p-AND-passing set and p-OR-passing set collapse to the passing set $\mathcal{A}_i(A)$, while the p-AND-rejection and p-OR-rejection sets collapse to the rejection set $\bar{\mathcal{A}}_i(A)$.

Apart from the above definitions, we need to define subsets of the above sets as follows,

The *p-AND-Truth* set (p-AND-strongly passing set)

$$\hat{T}_{ip}(A) \equiv \{ (a_{j(1)}, \ldots, a_{j(p)}) \in \hat{\mathcal{A}}_{ip}(A) \mid x_i(\mathrm{NOT}(a_{j(1)} \wedge \ldots \wedge a_{j(p)}) \mid A) = 0 \}$$

where $\mathrm{NOT}(a_{j(1)} \wedge \ldots \wedge a_{j(p)}) \in A$

This set contains only those groups of statements of A that belong to the p-AND-passing set of component i such that the negation of the conjunction of the statements in the group is rejected by i. In short, component i passes the statements individually and rejects the negation of their conjunction. In particular, when $p=1$, $\hat{T}_{ip}(A) = T_i(A)$ as expected.

Note

(1) If $x_i(\mathrm{NOT}(a_{j(1)} \wedge \ldots \wedge a_{j(p)}) \mid A) = 0$ and component i is (I consistent) then $x_i(\overline{b} \mid A) = 0$ for any $b \in A$ such that $(a_{j(1)} \wedge \ldots \wedge a_{j(p)}) \Rightarrow b$, because then $\overline{b} \Rightarrow \mathrm{NOT}(a_{j(1)} \wedge \ldots \wedge a_{j(p)})$ and by I consistency $x_i(\overline{b}) \leq x_i(\mathrm{NOT}(a_{j(1)} \wedge \ldots \wedge a_{j(p)}))$. This observation allows us to replace in the above definition the negation of the conjunction by any implicant of it as long as the component is I consistent. For example, $\mathrm{NOT}(a_1 \wedge a_2) \Rightarrow \mathrm{NOT}(a_{j(1)} \wedge \ldots \wedge a_{j(p)})$.

The *p-AND-Contradictory* set

$$\hat{C}_{ip}(A) \equiv \{ (a_{j(1)}, \ldots, a_{j(p)}) \in \hat{\mathcal{A}}_{ip}(A) \mid x_i(\mathrm{NOT}(a_{j(1)} \wedge \ldots \wedge a_{j(p)}) A) = 1 \}$$

where $\mathrm{NOT}(a_{j(1)} \wedge \ldots \wedge a_{j(p)}) \in A$.

This set contains those groups of statements from $\hat{\mathcal{A}}_{ip}(A)$ (the p-AND-passing set) such that the negation of the conjunction of the statements in the group is also passing. In short, component i passes the statements individually

but also passes the negation of their logic conjunction, thus the name "contradictory". When p=1, $\hat{C}_{ip}(A)=C_i(A)$. Again if x_i is I consistent then the requirement that $x_i(\mathrm{NOT}(a_{j(1)}\wedge\ldots\wedge a_{j(p)})|A)=1$ means that $x_i(\overline{b}|A)=1$ for any b such that $\mathrm{NOT}(a_{j(1)}\wedge\ldots\wedge a_{j(p)})=>\overline{b}$ or $b=>(a_{j(1)}\wedge\ldots\wedge a_{j(p)})$.

The *p-AND-Blocked* set

$$\hat{B}_{ip}(A)\equiv\{(a_{j(1)},\ldots,a_{j(p)})\in\hat{\hat{A}}_{ip}(A)\,|\,x_i(\mathrm{NOT}(a_{j(1)}\wedge\ldots\wedge a_{j(p)})\,|\,A)=0\}$$

where $\mathrm{NOT}(a_{j(1)}\wedge\ldots\wedge a_{j(p)})\in A)$.

This set contains those groups of statements from $\hat{\hat{A}}_{ip}$ (the p-AND-rejection set of component i) so that the negation of their AND conjunction is also rejected. In other words, it contains groups of p statements from A so that i rejects at least one of them, but also rejects the negation of their AND conjunction. For p=1, $\hat{B}_{ip}=B_i$.

The *p-AND-False* set (p-AND strongly rejection set)

$$\hat{F}_{ip}(A)\equiv\{(a_{j(1)},\ldots,a_{j(p)})\in\hat{\hat{A}}_{ip}(A)\,|\,x_i(\mathrm{NOT}(a_{j(1)}\wedge\ldots\wedge a_{j(p)})\,|A)=1\}$$

where $\mathrm{NOT}(a_{j(1)}\wedge\ldots\wedge a_{j(p)})\in A$.

This set contains those groups of p statements of A so that all p of them are rejected by component i when considered individually, but the negation of their AND conjunction is accepted. When p=1, $\hat{F}_{ip}=F_i$.

Similarly, we define the OR counterparts of the sets above:

The *p-OR-Truth* set (p-OR-strongly passing set)

$$\check{T}_{ip}(A) \equiv \{ (a_{j(1)}, \ldots, a_{j(p)}) \in \hat{\mathbb{A}}_{ip}(A) \mid x_i(\text{NOT}(a_{j(1)} \vee \ldots \vee a_{j(p)}) \mid A) = 0 \}$$

where $\text{NOT}(a_{j(1)} \vee \ldots \vee a_{j(p)}) \in A$).

It contains those groups of p statements of *A* (regardless of complexity) so that at least one of them is passing by component i when considered individually, but the negation of their OR disjunction is not passing.

The *p-OR-Contradictory* set

$$\check{C}_{ip}(A) \equiv \{ (a_{j(1)}, \ldots, a_{j(p)}) \in \check{\mathbb{A}}_{ip}(A) \mid x_i(\text{NOT}(a_{j(1)} \vee \ldots \vee a_{j(p)}) \mid A) = 1 \}$$

where $\text{NOT}(a_{j(1)} \vee \ldots \vee a_{j(p)}) \in A$.

It contains those groups of p statements of *A* (regardless of complexity) such that at least one is passing by component i when considered individually, but the negation of their OR disjunction is also passing.

The *p-OR-Blocked* set

$$\check{B}_{ip}(A) \equiv \{ (a_{j(1)}, \ldots, a_{j(p)}) \in \check{\bar{\mathbb{A}}}_{ip}(A) \mid x_i(\text{NOT}(a_{j(1)} \vee \ldots \vee a_{j(p)}) \mid A) = 0 \}$$

where $\text{NOT}(a_{j(1)} \vee \ldots \vee a_{j(p)}) \in A$.

It contains those groups of p statements of *A* whose OR complexity w.r.t. $\bar{\mathbb{A}}_i(A)$ is p and so that all of them are rejected when considered individually by component i, but also the negation of their OR disjunction is rejected.

The *p-OR-False* set (the p-OR strong rejection set),

$$\check{F}_{ip}(A) \equiv \{ (a_{j(1)}, \ldots, a_{j(p)}) \in \check{\check{\mathcal{A}}}_{ip}(A) \mid x_i(\mathrm{NOT}(a_{j(1)} \vee \ldots \vee a_{j(p)}) \mid A) = 1 \}$$

where $\mathrm{NOT}(a_{j(1)} \vee \ldots \vee a_{j(p)}) \in A$.

This set contains all groups of p statements of A whose OR complexity w.r.t. $\hat{\mathcal{A}}_{ip}$ is p and so that all of them are rejected by component i when considered individually , but the negation of their OR disjunction is passing.

Again for p=1

$\check{T}_{ip}=T_i=\hat{T}_{ip}$

$\check{C}_{ip}=C_i=\hat{C}_{ip}$

$\check{B}_{ip}=B_i=\hat{B}_{ip}$

$\check{F}_{ip}=F_i=\hat{F}_{ip}$

In general,

$$\hat{\mathcal{A}}_{ip} \supseteq \hat{T}_{ip} \cup \hat{C}_{ip} \tag{3.5}$$

$$\hat{\hat{\mathcal{A}}}_{ip} \supseteq \hat{F}_{ip} \cup \hat{B}_{ip} \tag{3.6}$$

$$\check{\mathcal{A}}_{ip} \supseteq \check{T}_{ip} \cup \check{C}_{ip} \tag{3.7}$$

$$\check{\check{\mathcal{A}}}_{ip} \supseteq \check{F}_{ip} \cup \check{B}_{ip} \tag{3.8}$$

Observe that,

$\hat{\mathcal{A}}_{ip}$, $\hat{\hat{\mathcal{A}}}_{ip}$ are mutually exclusive. The same holds for $\check{\mathcal{A}}_{ip}$, $\check{\check{\mathcal{A}}}_{ip}$. Further, $\hat{T}_{ip}$, $\hat{C}_{ip}$, $\hat{B}_{ip}$, $\hat{F}_{ip}$ are mutually exclusive and so are the sets $\check{T}_{ip}$, $\check{C}_{ip}$, $\check{B}_{ip}$, $\check{F}_{ip}$.

Pictorially, we have,

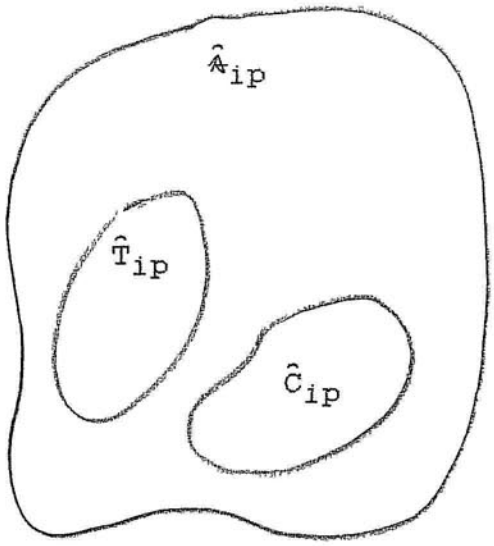

Fig. 3.7 The p-AND-passing set contains the p-AND-Truth set and the p-AND-Contradictory set

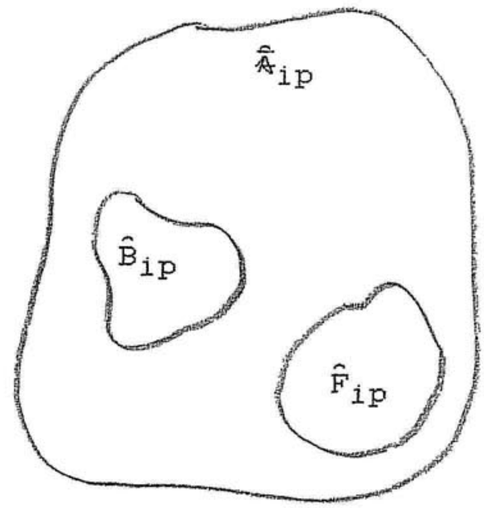

Fig. 3.8 The p-AND-rejection set contains the p-AND-Blocked set and the p-AND-False set

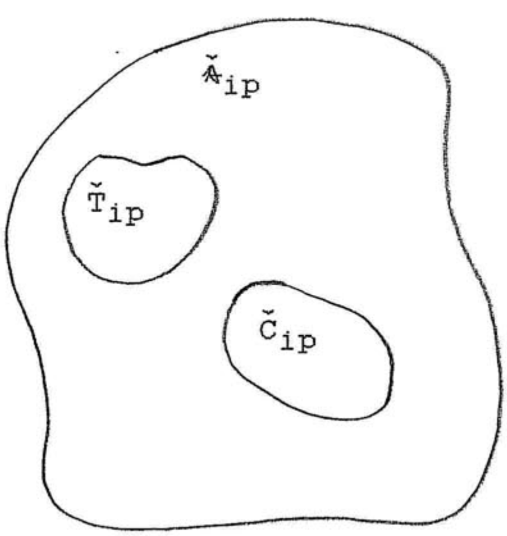

Fig. 3.9 The p-OR-passing set contains the p-OR-Truth set and the p-OR-Contradictory set

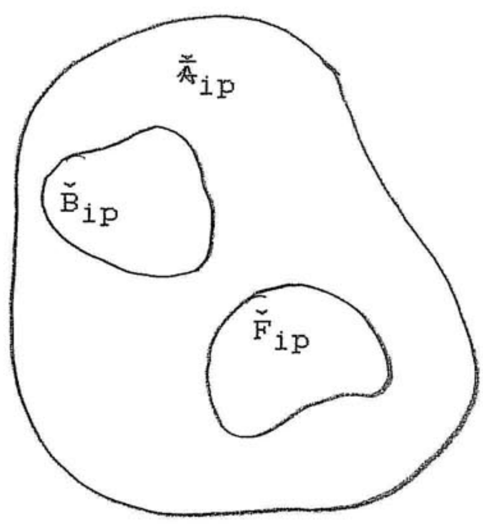

Fig. 3.10 The p-OR-rejection set contains the p-OR-Blocked set and the p-OR-False set

Note

If A is such that for each group of p statements the negation of their conjunction is also included in A, (this is the case for example when A is augmented to become $\mathcal{L}_{NOT}\mathcal{L}_{AND}[A]$) then $\hat{\mathcal{A}}_{ip}=\hat{T}_{ip}\cup\hat{C}_{ip}$. Similar observations hold for $\hat{\hat{\mathcal{A}}}_{ip}$, $\check{\mathcal{A}}_{ip}$, $\check{\check{\mathcal{A}}}_{ip}$. Again if A is augmented to A_H then all relations (3.5), (3.6), (3.7), (3.8) become equalities.

3.4 Effect of Logic Behaviour of Components on their Outcome Sets

At this point a question naturally arises: What is the effect of the logic behaviour of component i, as described in section 3.3, on the form of the outcome sets of component i. These effects are examined in the following propositions:

Proposition 3.6

The following statements hold,

(a) If x_i is L-NAND consistent then $\hat{C}_{ip}(A)=\emptyset$, for all A and for any p.

(b) If x_i is R-NAND consistent then $\hat{B}_{ip}(A)=\emptyset$, for all A and for any p.

Proof:

(a) Pick a group of p statements $(a_{j(1)},\dots,a_{j(p)})$ from A whose AND complexity w.r.t. $\mathcal{A}_i(A)$ is p and so that $\Pi^p_{K=1}x_i(a_{j(K)}|A)=1$ while $NOT(a_{j(1)}\wedge\dots\wedge a_{j(p)})\in A$. If no such group of statements exists, then $\hat{C}_{ip}(A)=\emptyset$ and the statement is trivially proved. Because of L-NAND consistency $\Pi^p_{K=1}x_i(a_{j(K)}|A)\leq 1-x_i(NOT(a_{j(1)}\wedge\dots\wedge a_{j(p)})|A)$ and thus $1-x_i(NOT(a_{j(1)}\wedge\dots\wedge a_{j(p)})|A)=1$. Hence, $x_i(NOT(a_{j(1)}\wedge\dots\wedge a_{j(p)})|A)=0$ which implies that $\hat{C}_{ip}(A)=\emptyset$.

(b) Pick a group of p statements $(a_{j(1)},\dots,a_{j(p)})$ from A so that $\Pi^p_{j=1}x_i(a_j|A)=0$ and $NOT(a_{j(1)}\wedge\dots\wedge a_{j(p)})\in A$. If no such group of statements exists then $\hat{B}_{ip}=\emptyset$ trivially. Now

because of R-NAND consistency

$\Pi^p_{k=1} x_i(a_{j(k)}) \geq 1 - x_i(\text{NOT}(a_{j(1)} \wedge \ldots \wedge a_{j(p)}) \mid A)$ and thus $x_i(\text{NOT}(a_{j(1)} \wedge \ldots \wedge a_{j(p)}) = 1$, hence, $\hat{B}_{ip} = \emptyset$. //

Proposition 3.7

The following statements hold:

(a) If x_i is L-NOR consistent then $\check{C}_{ip}(A) = \emptyset$ for all A and for any p.

(b) If x_i is R-NOR consistent then $\check{B}_{ip}(A) = \emptyset$ for all A and for any p.

Proof:

(a) Pick a group of p statements from A, say, $(a_{j(1)}, \ldots, a_{j(p)})$ and so that $\sqcup^p_{k=1} x_i(a_{j(k)} \mid A) = 1$ and $\text{NOT}(a_{j(1)} \vee \ldots \vee a_{j(p)}) \in A$. If no such group exists then $\check{C}_{ip} = \emptyset$ trivially. Because of L-NOR consistency $1 = \sqcup^p_{k=1} x_i(a_{j(k)} \mid A) \leq 1 - x_i(\text{NOT}(a_{j(1)} \vee \ldots \vee a_{j(p)}) \mid A)$, hence, $x_i(\text{NOT}(a_{j(1)} \vee \ldots \vee a_{j(p)}) \mid A) = 0$ and therefore, $\check{C}_{ip}(A) = \emptyset$.

(b) Using the fact that, $0 = \sqcup^p_{k=1} x_i(a_{j(k)} \mid A) \geq 1 - x_i(\text{NOT}(a_{j(1)} \vee \ldots \vee a_{j(p)}) \mid A)$, it follows that $x_i(\text{NOT}(a_{j(1)} \vee \ldots \vee a_{j(p)})) = 1$ or $\check{B}_{ip} = \emptyset$. //

Proposition 3.8

If x_i is AND, NOT, OR consistent or equivalently NAND consistent or NOR consistent then,

$$\hat{C}_{ip} = \hat{B}_{ip} = \check{C}_{ip} = \check{B}_{ip} = \emptyset$$

Proof:

By Proposition 3.5 we know that,

NAND<=>NOR<=>[AND, NOT]<=>[OR, NOT].

Now combine Propositions 3.6 and 3.7. //

Propositions 3,6, 3.7, 3.8 can be summarized in the following table,

	L-NAND	R-NAND	L-NOR	R-NOR	NAND*
$\hat{T}_{ip}$					
$\hat{C}_{ip}$	$\emptyset$				$\emptyset$
$\hat{B}_{ip}$		$\emptyset$			$\emptyset$
$\hat{F}_{ip}$					
$\check{T}_{ip}$					
$\check{C}_{ip}$			$\emptyset$		$\emptyset$
$\check{B}_{ip}$				$\emptyset$	$\emptyset$
$\check{F}_{ip}$					

Fig. 3.11 Some outcome sets are empty for all A and p depending on the logic behaviour of the component.

(*) Recall that NAND consistency is equivalent to NOR consistency and in turn to [AND, NOT] and to [OR, NOT].

Chapter 4

LOGIC ON COHERENT STRUCTURES

Once we are given the rules of logic for statements and also the logic behaviour of components, what do we expect or can we impose on the logic behaviour of coherent structures whose components obey the logic properties that we examined in previous chapters? Roughly speaking we would like the structures to keep some of the logic properties of their components. In fact, this proves to be true to a certain extend.

It is shown that I consistency (i.e. Implication consistency) is respected by coherent structures whose components are I consistent.

Further, L-NAND consistency and R-NOR consistency are preserved at the structure level (when components respect it) by some particular classes of coherent structures. Namely, $\mathcal{S}_p$ and

$\overline{S}_p$.

As a consequence it is shown that S_p structures are the only structures which, when their components are L-NAND consistent, they will never be contradictory for statements whose AND complexity does not exceed p-1. A similar theorem holds for L-NOR consistent components in S_2 structures where it is shown that S_2 are the only structures that are never contradictory under OR disjunctions of statements.

Also, symmetrical results are shown for $\overline{S}_p$, $\overline{S}_2$ structures.

Finally, it is shown that the only structures that are NOT consistent when their components are so, are those in the class M (self dual structures).

4.1 Outcome Sets

Any coherent structure φ, viewed as a function of the statements, $\varphi(\mathbf{x}(.))$ is binary (i.e. it is 1 if statement a_j passes and it is 0 if a_j is rejected). Also, $\varphi(\mathbf{x}(t|A))=1$ and $\varphi(\mathbf{x}(f|A))=0$ because all components are equal to 1 when t is proposed to them (they agree to pass the i-true statement) and φ is coherent, while they are all 0 when f is presented to them. (They agree to reject the i-false statement). Therefore, all definitions and properties on logic consistencies (I consistency, NAND etc.) apply for structures as well. The same holds for the outcome sets of the structures,

$\hat{T}_{\varphi p}$, $\hat{C}_{\varphi p}$, $\hat{B}_{\varphi p}$, $\hat{F}_{\varphi p}$;

$\check{T}_{\varphi p}$, $\check{C}_{\varphi p}$, $\check{B}_{\varphi p}$, $\check{F}_{\varphi p}$.

For example, we may recall that,

the *passing* set of φ is

$\mathcal{A}_{\varphi}(A)=\{a_j | a_j \in A,\ \varphi(\mathbf{x}(a_j|A))=1\}$

the *p-AND-passing* set of φ is

$\hat{\mathcal{A}}_{\varphi p}(A)=\{(a_{j(1)},,,a_{j(p)}) | \Pi^{p}_{K=1}\varphi(\mathbf{x}(a_{j(K)}|A)$
$)=1,\ C_{AND}(a_{j(1)}\wedge\ldots\wedge a_{j(p)}|\mathcal{A}_{\varphi}(A))=p\}$

the *p-AND-truth* set of φ is

$\hat{T}_{\varphi p}(A)=\{(a_{j(1)},\ldots,a_{j(p)})\in\hat{\mathcal{A}}_{\varphi p}(A) |$
$\varphi(\mathbf{x}(NOT(a_{j(1)}\wedge\ldots\wedge a_{j(p)})|A)=0\}$

where $NOT(a_{j(1)}\wedge\ldots\wedge a_{j(p)})\in A$.

Remark

(a) If components in φ are NAND then

$\varphi(\mathbf{x}(\mathrm{NOT}(a_{j(1)}\wedge\ldots\wedge a_{j(p)})\,|A)=$

$=\varphi(\mathbf{1}-\Pi^{p}_{k=1}\mathbf{x}(a_{j(k)}\,|A))=$

$=1-\varphi^{D}(\Pi^{p}_{k=1}\mathbf{x}(a_{j(k)}\,|A))$

Similar simplifications hold respectively for the quantities appearing in other outcome sets.

(b) If components are "absolute" or if $A=A_H$ then $\hat{A}_{\varphi p}=\hat{T}_{\varphi p}\cup\hat{C}_{\varphi p}$ exactly as in the case of single components. Similar observations hold for the other outcome sets as well.

Depending on the properties of the structure φ and the logic behaviour of its components, x_i's, the outcome sets of $\varphi(\mathbf{x}(.))$ have to obey certain properties which we propose to explore below.

Proposition 4.1 (Preservation of I consistency)

A coherent structure φ is I consistent if its components are I consistent.

Proof:

Let the components be I consistent. Then,

$$\mathbf{x}(a_1)\geq\mathbf{x}(a_1)\mathbf{x}(a_2) \tag{4.1}$$

Since φ is coherent we obtain,

$$\varphi(\mathbf{x}(a_1))\geq\varphi(\mathbf{x}(a_1)\mathbf{x}(a_2)) \tag{4.2}$$

Also

$$\varphi(\mathbf{x}(a_2))\geq\varphi(\mathbf{x}(a_1)\mathbf{x}(a_2)) \tag{4.3}$$

From (4.2), (4.3) we obtain,

$$\varphi(\mathbf{x}(a_1))\varphi(\mathbf{x}(a_2))\geq\varphi(\mathbf{x}(a_1)\mathbf{x}(a_2)) \tag{4.4}$$

By I consistency of components,

$$\mathbf{x}(a_1)\,\mathbf{x}(a_2) \geq \mathbf{x}(a_1 \wedge a_2) \tag{4.5}$$

But since φ is coherent,

$$\varphi(\mathbf{x}(a_1)\,\mathbf{x}(a_2)) \geq \varphi(\mathbf{x}(a_1 \wedge a_2)) \tag{4.6}$$

From (4.4) and (4.6) we obtain,

$$\varphi(\mathbf{x}(a_1))\,\varphi(\mathbf{x}(a_2)) \geq \varphi(\mathbf{x}(a_1 \wedge a_2)) \tag{4.7}$$

which is the condition for I consistency of φ. //

Note that we use the notation $\mathbf{x}(a)$ instead of $\mathbf{x}(a|A)$ for simplicity.

Depending on the number of statements, $|\mathcal{A}_i|$, that each component i of the structure φ passes, and on the logic behaviour of the components, limitations are implied on the outcome sets of the coherent structure φ.

Proposition 4.2

Let the components of structure φ be R-NAND consistent and suppose that $|\mathcal{A}_i| \leq p$ for all i then,

(a) $\hat{T}_{\varphi q}=\emptyset$, $\hat{B}_{\varphi q}=\emptyset$ for all $q \geq p+1$

(b) $|\hat{T}_{\varphi p}| \leq 1$

(c) $\hat{T}_{\varphi p} \neq \emptyset \Rightarrow \hat{T}_{\varphi p}=\hat{\mathcal{A}}_{\varphi p}$ and $\hat{B}_{\varphi p}=\emptyset$

(d) $\hat{B}_{\varphi p} \neq \emptyset \Rightarrow \hat{\mathcal{A}}_{\varphi p}=\emptyset$

Proof:

(a) Let to the contrary a group of statements $(a_1,\ldots,a_q) \in \hat{T}_{\varphi q}$ (or $(a_1,\ldots,a_q) \in \hat{B}_{\varphi q}$) with $q \geq p+1$. Then by definition of $\hat{T}_{\varphi q}$ ($\hat{B}_{\varphi q}$), $\varphi(\mathbf{x}(\mathrm{NOT}(a_1,\ldots,a_q)))=0$. It follows that there is a min cut in φ for which all the components in it have $x_i(\mathrm{NOT}(a_1,\ldots,a_q))=0$. But components are R-

NAND consistent, therefore,

$$\Pi^{q}_{j=1} x_i(a_j) \geq 1 - x_i(\mathrm{NOT}(a_1, \ldots, a_q)) = 1$$

for each component i in the cut. Thus ,

$$\Pi^{q}_{j=1} x_i(a_j) = 1$$

and therefore, component i passes $a_1, \ldots, a_q$. But $q \geq p+1$ and thus $|\mathcal{A}_i| = q \geq p+1$. Contradiction.

(b) Let to the contrary $(a_1, \ldots, a_p)$ and $(b_1, \ldots, b_p)$ both belong to $\hat{T}_{\varphi p}$ and they are not identical. So assume some elements to be different between the two groups:

$$\{a_1, \ldots, a_p\} \cap \{b_1, \ldots, b_p\} = \{a_1, \ldots, a_K\} = \{b_1, \ldots, b_K\} \text{ with } 0 \leq K < p$$

Now,

$$\Pi^{p}_{j=1} \varphi(\mathbf{x}(a_j)) = 1 \text{ and } \varphi(\mathbf{x}(\mathrm{NOT}(a_1 \wedge \ldots \wedge a_p))) = 0$$

It follows that for each a_j there is a min path that passes it and a min cut where all the components satisfy $x_i(\mathrm{NOT}(a_1 \wedge \ldots \wedge a_p)) = 0$. Because of R-NAND consistency of components,

$$\Pi^{p}_{j=1} x_i(a_j) \geq 1 - x_i(\mathrm{NOT}(a_1 \wedge \ldots \wedge a_p)) = 1$$

for all components in the cut. This implies that $x_i(a_j) = 1$ for $j = 1, \ldots, p$ and for all i in the cut. It follows that each min path contains at least one component that passes all of the statements $a_1, \ldots, a_p$.

The same arguments hold for $b_1, \ldots, b_p$.

Take now a min path that passes b_{K+1}, which is not common between the two groups of statements. This min path must contain a component i who passes $a_1, \ldots, a_p, b_{K+1}$. Then it must pass p+1 different statements and therefore $|\mathcal{A}_i| \geq p+1$ which is a contradiction.

(c) Case 1: $\hat{T}_{\varphi p} \neq \emptyset \Rightarrow \hat{T}_{\varphi p} = \hat{\mathcal{A}}_{\varphi p}$

Let $(a_1, \ldots, a_p) \in \hat{T}_{\varphi p}$. Then there is a cut for which each component has $x_i(\text{NOT}(a_1 \wedge \ldots \wedge a_p)) = 0$ and because of R-NAND consistency of components, each one in the cut must pass all of the statements $a_1, \ldots, a_p$. Thus, each min path of φ contains at least one component that passes all of the statements $a_1, \ldots, a_p$. Now suppose to the contrary that $\hat{T}_{\varphi p} \neq \hat{A}_{\varphi p}$ then there is a group of statements $(c_1, \ldots, c_p) \in \hat{A}_{\varphi p} - \hat{T}_{\varphi p}$. Then for each c_j, there is a min path of φ that passes it. Let c_{K+1} be a statement not common between $(a_1, \ldots, a_p)$ and $(c_1, \ldots c_p)$. Take the min path of φ that passes c_{K+1}. In this min path there is a component i that passes $a_1, \ldots, a_p, c_{K+1}$. Thus, it passes p+1 statements. Contradiction. Therefore, $\hat{T}_{\varphi p} = \hat{A}_{\varphi p}$.

Case 2: $\hat{T}_{\varphi p} \neq \emptyset \Rightarrow \hat{B}_{\varphi p} = \emptyset$.

Since $\hat{T}_{\varphi p} \neq \emptyset$, there is $(a_1, \ldots, a_p) \in \hat{T}_{\varphi p}$. For each statement a_j there is a min path of φ that passes it. Further, by arguing as above, R-NAND consistency and the fact that $x_i(\text{NOT}(a_1 \wedge \ldots \wedge a_p)) = 0$ for each i in some min cut of φ, imply that in each min path of φ there is a component that passes all of $a_1, \ldots, a_p$.

Now suppose to the contrary that there is $(b_1, \ldots, b_p) \in \hat{B}_{\varphi p}$ and $(b_1, \ldots, b_p) \neq (a_1, \ldots, a_p)$ then $\varphi(\mathbf{x}(\text{NOT}(b_1 \wedge \ldots \wedge b_p))) = 0$ and there is a min cut in φ that has $x_i(\text{NOT}(b_1 \wedge \ldots \wedge b_p)) = 0$ for all i in the cut. Again, by R-NAND consistency all components in the cut must pass each of the statements $b_1, \ldots, b_p$ which implies that each min path has a component that passes all of $b_1, \ldots, b_p$. Take the min path that passes a_K which is not common between $(a_1, \ldots, a_p)$ and $(b_1, \ldots, b_p)$. In that min path there is a com-

ponent that passes $b_1, \ldots, b_p, a_K$. Thus $|\mathcal{A}_i| \geq p+1$. Contradiction.

(d) $\hat{B}_{\varphi p} \neq \emptyset \Rightarrow \hat{\mathcal{A}}_{\varphi p} = \emptyset$

$\hat{B}_{\varphi p} \neq \emptyset \Rightarrow$ there is $(a_1, \ldots, a_p) \in \hat{B}_{\varphi p}$, thus there is a cut of φ where each component i has $x_i(\mathrm{NOT}(a_1 \wedge \ldots \wedge a_p)) = 0$ and by R-NAND consistency there is a component in each path that passes all of $a_1, \ldots, a_p$.

Let now to the contrary $(b_1, \ldots, b_p) \in \hat{\mathcal{A}}_{\varphi p}$. Then there is a min path that passes b_K which is not common with $(a_1, \ldots, a_p)$. Then in that min path there is a component that passes $a_1, \ldots, a_p, b_K$. Thus $|\mathcal{A}_i| \geq p+1$. Contradiction. //

Remark

Observe that when $\hat{T}_{\varphi p} \neq \emptyset$ then when components are I consistent $\hat{T}_{\varphi q} \neq \emptyset$ for all $q \leq p-1$. This holds because $x_i(\mathrm{NOT}(a_1 \wedge \ldots \wedge a_p)) = 0$ and by implication consistency $x_i(\mathrm{NOT}(a_1 \wedge \ldots \wedge a_{p-1})) = 0$.

Corollary 4.3

If the components of a structure φ are R-NAND consistent and for each component i $|\mathcal{A}_i| \leq p$, then exactly one of the following holds:

(a) $|\hat{T}_{\varphi p}| = 1$, $\hat{T}_{\varphi p} = \hat{\mathcal{A}}_{\varphi p}$, $\hat{C}_{\varphi p} = \emptyset$,
$\hat{B}_{\varphi p} = \emptyset$.

(b) $\hat{T}_{\varphi p} = \emptyset$, $\hat{B}_{\varphi p} = \emptyset$, $\hat{\mathcal{A}}_{\varphi p} \neq \emptyset$

(c) $\hat{B}_{\varphi p} \neq \emptyset$, $\hat{\mathcal{A}}_{\varphi p} = \emptyset$ (and hence $\hat{T}_{\varphi p} = \emptyset$)

Proof:

By Proposition 4.2. //

Proposition 4.4

Let the components of structure φ be L-NOR consistent and suppose that $|\bar{\mathcal{A}}_i| \leq p$ for all components i of the structure, then,

(a) $\check{F}_{\varphi q}=\emptyset$, $\check{C}_{\varphi q}=\emptyset$ for $q \geq p+1$

(b) $|\check{F}_{\varphi q}| \leq 1$

(c) $\check{F}_{\varphi p} \neq \emptyset \Rightarrow \check{F}_{\varphi p}=\check{\bar{\mathcal{A}}}_{\varphi p}$ and $\check{C}_{\varphi p}=\emptyset$

(d) $\check{C}_{\varphi p} \neq \emptyset \Rightarrow \check{\bar{\mathcal{A}}}_{\varphi p}=\emptyset$.

Proof:

The arguments are the same as in Proposition 4.2 using now paths instead of cuts. We will prove (a) for example,

(a) $\check{F}_{\varphi q} \neq \emptyset \Rightarrow$ there is $a_1, \ldots, a_p$ so that $\varphi(\mathbf{x}(\mathrm{NOT}(a_1 \vee \ldots \vee a_p)))=1$. Then there is a min path of φ for which all the components satisfy $x_i(\mathrm{NOT}(a_1 \vee \ldots \vee a_p))=1$. By L-NOR consistency of components $\sqcup^q{}_{j=1} x_i(a_j) \leq 1-x_i(\mathrm{NOT}(a_1 \vee \ldots \vee a_p))=0$. Thus, $|\bar{\mathcal{A}}_i|=q \geq p+1$. Contradiction.

(b), (c) and (d) are proved similarly. //

Corollary 4.5

If the components of a structure φ are L-NOR consistent and for each component i $|\bar{\mathcal{A}}_i| \leq p$, then exactly one of the following holds:

(a) $|\check{F}_{\varphi p}|=1$, $\check{F}_{\varphi p}=\check{\mathcal{A}}_{\varphi p}$, $\check{B}_{\varphi p}=\emptyset$,
$\check{C}_{\varphi p}=\emptyset$

(b) $\check{F}_{\varphi p}=\emptyset$, $\check{C}_{\varphi p}=\emptyset$, $\check{\bar{\mathcal{A}}}_{\varphi p} \neq \emptyset$

(c) $\check{C}_{\varphi p} \neq \emptyset$, $\check{\bar{\mathcal{A}}}_{\varphi p}=\emptyset$, $\check{F}_{\varphi p}=\emptyset$.

Proof:

Directly from Proposition 4.4. //

Corollary 4.6

If the components of a structure are R-NOT consistent and $|\mathcal{A}_i|=1$ for all components in the structure then,

(a) $|T_\varphi|\leq 1$

(b) $T_\varphi\neq\emptyset \Rightarrow T_\varphi=\mathcal{A}_\varphi$ and $B_\varphi=\emptyset$

(c) $B_\varphi\neq\emptyset \Rightarrow \mathcal{A}_\varphi=\emptyset$

Proof:

R-NAND consistency for p=1 becomes R-NOT consistency and using Proposition 4.2 we obtain the result. //

Corollary 4.7

If the components of a structure φ are L-NOT consistent and $|\overline{\mathcal{A}}_i|=1$ for all components then,

(a) $|F_\varphi|\leq 1$

(b) $F_\varphi\neq\emptyset \Rightarrow C_\varphi=\emptyset$, $F_\varphi=\overline{\mathcal{A}}_\varphi$

(c) $C_\varphi\neq\emptyset \Rightarrow \overline{\mathcal{A}}_\varphi=\emptyset$.

Proof:

L-NOR implies L-NOT for p=1. Then use Proposition 4.4 //

Observe that if *A* contains two statements and each component is obliged to pass one and reject the other, $|\mathcal{A}_i|=|\overline{\mathcal{A}}_i|=1$, then assuming that components are NOT consistent, one of the following holds,

(a) T_φ, F_φ have one element each and the rest of the out-

come sets are empty.

(b) $B_\varphi \neq \emptyset$ and all other outcome sets are empty.

(c) $C_\varphi \neq \emptyset$ and all other outcome sets are empty.

4.2 Logic Behaviour of $\mathcal{S}_p$ Structures

Structures in $\mathcal{S}_p$ have at least one common component among any p of its min paths. This property, along with assumptions on the logic behaviour of the components, imply properties on the behaviour of the structure as a whole.

Proposition 4.8 (Closure of L-NAND for structures in $\mathcal{S}_p$)
If $\varphi \in \mathcal{S}_p$ and the components are L-NAND consistent then $\varphi(\mathbf{x}(.))$ is L-NAND consistent for any group of p-1 statements $(a_1, \ldots, a_{p-1})$

Proof:

We want to show that,

$$\Pi^{p-1}{}_{j=1}\varphi(\mathbf{x}(a_j|A)) \leq 1-\varphi(\mathbf{x}(\mathrm{NOT}(a_1\wedge\ldots\wedge a_{p-1}|A))) \tag{4.8}$$

If $\Pi^{p-1}{}_{j=1}\varphi(\mathbf{x}(a_j))=0$ then (4.8) holds.
If $\Pi^{p-1}{}_{j=1}\varphi(\mathbf{x}(a_j))=1$ then $\varphi(\mathbf{x}(a_j))=1$ for $j=1,\ldots,p-1$
Then there are at most p-1 min paths that pass statements $a_1,\ldots,a_{p-1}$. Suppose now, to the contrary, that there is a min path that passes $\mathrm{NOT}(a_1\wedge\ldots\wedge a_{p-1})$. Since $\varphi \in \mathcal{S}_p$, any p min paths have a common component; therefore, there is a component, say i, for which $x_i(a_j)=1$, $j=1,\ldots,p-1$ and $x_i(\mathrm{NOT}(a_1\wedge\ldots\wedge a_{p-1}))=1$. But $x_i(.)$ is L-NAND consistent. Contradiction. //

Note

In this proof we implicitly assumed that φ has no less than p

min paths. In case φ has $q<p$ min paths we argue as follows: Since $\varphi \in \mathcal{S}_p$, by definition of $\mathcal{S}_p$ it follows that $\varphi \in \mathcal{S}_q$. The p-1 statements of $\mathcal{A}_\varphi(A)$ forming $a_1 \wedge \ldots \wedge a_{p-1}$ will need at most q min paths to pass φ. Suppose there is a min path that passes $\mathrm{NOT}(a_1 \wedge \ldots \wedge a_{p-1})$. Since any q min paths have a common component, the arguments of the proof are repeated i.e. there will be a common component that violates L-NAND consistency which contradicts our assumption. //

Discussion

(1) It is clear that Proposition 4.8 will continue to hold if the components satisfy a stronger property, like NAND consistency. However, we gain no more than L-NAND consistency at the structure level.

Proposition 4.8 is interesting because it assures us that if we form structures of structures (i.e. structures whose components are structures themselves), the overall structure will satisfy L-NAND consistency whenever the individual structures do. This was expected since $\mathcal{S}_p$ structures whose components are $\mathcal{S}_p$ structures are again $\mathcal{S}_p$ structures; since any p min paths will still have at least on common component. It is also clear that if an $\mathcal{S}_p$ structure has components which are $\mathcal{S}_{p(1)}, \ldots, \mathcal{S}_{p(n)}$ structures, then the overall structure will behave as an $\mathcal{S}_q$ structure where $q=\min\{p_1, \ldots, p_n\}$.

(2) Still a question remains: What happens if we demand AND consistency at the structure level? Is this reasonable? Unfortunately, if AND consistency is demanded at the structure level it means that whenever $\varphi(\mathbf{x}(a_1))=1$ and $\varphi(\mathbf{x}(a_2))=1$

then we want $\varphi(\mathbf{x}(a_1 \wedge a_2))=1$. But this implies that for any two alternatives that pass the structure there is a min path that passes their conjunction. This must be true for three, four, etc. alternatives. But then the structure must be a series structure which is very limiting indeed. This is the reason for defining and using other types of logic consistencies like NAND, L-NAND, R-NAND, and their OR counterparts. Recall also the discussion at the end of section 1.2 where the same notion was met before the introduction of statements.

Corollary 4.9

Let $\varphi \in \mathcal{S}_p$ and let the components of φ be L-NAND consistent, then $\hat{C}_{\varphi p-1}=\emptyset$

Proof:

By Proposition 4.8 φ is L-NAND consistent for any group of p-1 statements. Therefore,

$$\Pi^{p-1}{}_{j=1}\varphi(\mathbf{x}(a_j)) \leq 1-\varphi(\mathbf{x}(\mathrm{NOT}(a_1 \wedge \ldots \wedge a_{p-1})))=1$$

It follows that when $\Pi^{p-1}{}_{j=1}\varphi(\mathbf{x}(a_j))=1$ then $\varphi(\mathbf{x}(\mathrm{NOT}(a_1 \wedge \ldots \wedge a_{p-1})))=0$. Thus, $\hat{C}_{\varphi p-1}=\emptyset$ //

Corollary 4.9 can be strengthened,

Corollary 4.10

Let $\varphi \in \mathcal{S}_p$ and let the components of φ be L-NAND consistent, then $\hat{C}_{\varphi q}=\emptyset$, for all $q \leq p-1$.

Proof:

$\varphi \in \mathcal{S}_p \Rightarrow \varphi \in \mathcal{S}_{p-1}$ and L-NAND consistency implies

$\hat{C}_{\varphi p-2}=\emptyset$ by Corollary 4.9. Now repeat the argument for p-2, p-3 etc. //

Proposition 4.8 can be strengthened in the Theorem below which says that the only structures that are never contradictory under conjunction of statements whose AND complexity w.r.t. the passing set $\mathcal{A}_\varphi$ is p-1, are the structures for which any p min paths have at least one common component and whose components are L-NAND consistent.

Theorem 4.11

Let a structure φ and let A be a set of statements so that $C_{AND}(\mathcal{A}_\varphi(A))\leq p-1$, then a necessary and sufficient condition for $\hat{C}_{\varphi q}(A)=\emptyset$ for all q for any A, is that $\varphi\in\mathcal{S}_p$ and its components are L-NAND consistent.

Proof:

First observe that $C_{AND}(\mathcal{A}_\varphi)\leq p-1$ implies that for all statements $b\in\mathcal{L}_{AND}[\mathcal{A}_\varphi]$ their complexity $C_{AND}(b|\mathcal{A}_\varphi)\leq p-1$.

(i) Sufficiency: Let $\varphi\in\mathcal{S}_p$ and the components L-NAND, then by Corollary 4.10 $\hat{C}_{\varphi q}=\emptyset$ for $q\leq p-1$. Also $\hat{C}_{\varphi q}=\emptyset$ for $q\geq p$ because $C_{AND}(\mathcal{A}_\varphi)\leq p-1$, while the definition of $\hat{C}_{\varphi q}$ requires the existence of a statement of complexity q. Hence, $\hat{C}_{\varphi q}=\emptyset$ for all q and for all A.

(ii) Necessity: Let $\varphi\notin\mathcal{S}_p$ but instead $\varphi\in\mathcal{S}_q$ where $1\leq q\leq p-1$. Without loss of generality let $C_{AND}(\mathcal{A}_\varphi)=p-1$. Choose $x_i(.)$'s so that p-1 min paths of φ pass $a_1,\ldots,a_{p-1}\in\mathcal{A}_\varphi$ (one statement for each min path) and

another min path that passes NOT$(a_1 \wedge \ldots \wedge a_{p-1})$. This is possible since only up to q min paths have a common component and thus no component violates his L-NAND consistency. It follows that $\hat{C}_{\varphi q} \neq \emptyset$. Contradiction. Therefore, $\varphi \in \mathcal{S}_p$.

It remains to show that the components must be L-NAND. Suppose that the components are not L-NAND. Since $\varphi \in \mathcal{S}_p$ and $C_{AND}(\mathcal{A}_\varphi) = p-1$, I choose $x_i(.)$'s so that for $a_j \in \mathcal{A}_\varphi$, $j=1,\ldots,p-1$ there are p-1 min paths of φ that pass $a_1,\ldots,a_{p-1}$ (one min path for each statement) and another path that passes NOT$(a_1 \wedge \ldots \wedge a_{p-1})$ (the common component among those p min paths is not L-NAND to prohibit this situation). Therefore, $\hat{C}_{\varphi q} \neq \emptyset$. Contradiction. Therefore, the components must be L-NAND. //

Remarks

(1) In the proof above it was assumed that $\varphi \in \mathcal{S}_p$ has at least p min paths . If instead it has less than p paths, say m, then the requirement $\varphi \in \mathcal{S}_p$ implies that all min paths of φ have a common component. (There is a one component cut). Then repeat the arguments using $a_1,\ldots,a_{m-1}$ statements from $\mathcal{A}_\varphi$.

(2) When p=1, L-NAND is equivalent to L-NOT and the results for $\mathcal{S}_p$ structures hold for $\mathcal{S}_2$ and L-NOT consistency.

(3) If A is acyclic w.r.t. AND operations, (recall that this means that we cannot construct i-false statements by AND conjunction of statements in A) then again $\mathcal{A}_\varphi$ is acyclic and thus $\hat{C}_{\varphi q} = \emptyset$ for all q without any restriction on the form of the structure φ. The problem in this case is trivial.

(4) Let us describe in words what was achieved by Theorem 4.11. If no restrictions are imposed on φ and A is allowed to be cyclic, then $\mathcal{A}_\varphi$ will in general be cyclic. Our aim is to impose conditions to assure that $\mathcal{A}_\varphi$ is acyclic w.r.t. AND conjunction. We look , therefore, at all statements that are constructed by AND conjunction of statements in $\mathcal{A}_\varphi$ and which are not cyclic, and demand that their negation (or in general an implicant of it when components obey I consistency) does not belong to $\mathcal{A}_\varphi$. The requirement that $\varphi \in \mathcal{S}_p$ and the components are L-NAND makes sure that the conjunction of any group of p statements or less in $\mathcal{A}_\varphi$ is not i-false (or that the negation of the conjunction of any p-1 statements in $\mathcal{A}_\varphi$ is not included in $\mathcal{A}_\varphi$). But this is not enough, as it does not tell us what happens for groups of more than p statements in $\mathcal{A}_\varphi$. By assuming that $C_{\mathrm{AND}}(\mathcal{A}_\varphi) \leq p-1$ we make sure that any acyclic conjunction of statements in $\mathcal{A}_\varphi$ can be formed by at most p-1 statements in $\mathcal{A}_\varphi$ and thus, we do not need to worry for groups of more than p-1 statements. Then we are sure that $\hat{C}_{\varphi q}=\emptyset$ for all q.

(5) Theorem 4.11 is interesting because it relates the form of the coherent structure, the logic complexity of the passing set of statements, the logic behaviour of the components and the logic behaviour of the structure as a whole.

A requirement of Theorem 4.11 is that $C_{\mathrm{AND}}(\mathcal{A}_\varphi) \leq p-1$. One way to assure this is to limit the number of statements that $\mathcal{A}_\varphi$ can contain; and to achieve this we may restrict the passing set of each component i, $\mathcal{A}_i(A)$, to contain at

most a certain number of elements. These thoughts are explored in the propositions below.

Proposition 4.12

If $\varphi \in \mathcal{S}_p$ and $|\mathcal{A}_i| \leq p-1$ for all i, then $|\mathcal{A}_\varphi| \leq p-1$.

Proof:

Suppose to the contrary that there are p or more elements in $\mathcal{A}_\varphi$. Then pick any p. There are at most p min paths of φ that pass these statements. However, any p paths have a common component since $\varphi \in \mathcal{S}_p$. This implies that the common component, say i, has $|\mathcal{A}_i| \geq p$. Contradiction. //

Proposition 4.13

If $|\mathcal{A}_\varphi| \leq p-1$ then $C_{AND}(\mathcal{A}_\varphi) \leq p-1$

Proof:

In general, for any set of statements B it is true that $|B| \geq C_{AND}(B)$, because the most complex AND statement that can be constructed from elements in B cannot contain more than B statements. //

Theorem 4.14

Let a coherent structure and A a set of statements, and $\max_i\{|\mathcal{A}_i(A)|\} \leq p-1$ then a necessary and sufficient condition for $\hat{C}_{\varphi q} = \emptyset$ for all A and all q is that $\varphi \in \mathcal{S}_p$ and components are L-NAND.

Proof:

Because of Proposition 4.12, 4.13, $\max|\mathcal{A}_i| \leq p-1 \Rightarrow C_{AND}(\mathcal{A}_\varphi) \leq p-1$. Then using Theorem 4.11 the result

obtains. //

Up to now we examined the behaviour of structures in $\mathcal{S}_p$ when components are L-NAND. It is time we are concerned with the properties of $\mathcal{S}_2$ structures ($\mathcal{S}_2 \supset \mathcal{S}_p$, $p \geq 3$) whose components are L-NOR.

Proposition 4.15 (Closure of L-NOR under $\mathcal{S}_2$)
If the components of a structure φ are L-NOR and $\varphi \in \mathcal{S}_2$ then φ is L-NOR.

Proof:
We want to prove that,

$$\sqcup^p{}_{j=1}\varphi(\mathbf{x}(a_j)) \leq 1-\varphi(\mathbf{x}(\mathrm{NOT}(a_1 \vee \ldots \vee a_p))) \qquad (4.9)$$

where $a_1, \ldots, a_p \in A$.
If $\sqcup^p{}_{j=1}\varphi(\mathbf{x}(a_j))=0$ then (4.9) trivially holds.
If $\sqcup^p{}_{j=1}\varphi(\mathbf{x}(a_j))=1$ then look at φ. $\varphi \in \mathcal{S}_2 \Rightarrow$
$\sqcup^p{}_{j=1}\varphi(\mathbf{x}(a_j))\varphi(\mathbf{1}-\sqcup^p{}_{j=1}\mathbf{x}(a_j))=0$ by Proposition 1.11. Since the first term is equal to 1, the second must equal zero:

$$\varphi(\mathbf{1}-\sqcup^p{}_{j=1}\mathbf{x}(a_j))=0 \qquad (4.10)$$

Components are L-NOR, therefore,

$$\sqcup^p{}_{j=1}x_i(a_j) \leq 1-x_i(\mathrm{NOT}(a_1 \vee \ldots \vee a_p))$$

or

$$1-\sqcup^p{}_{j=1}x_i(a_j) \geq x_i(\mathrm{NOT}(a_1 \vee \ldots \vee a_p)) \qquad (4.11)$$

Since $\varphi \in \mathcal{S}_2$ and φ is coherent then,

$$\varphi(\mathbf{1}-\sqcup^p{}_{j=1}\mathbf{x}_i(a_j)) \geq \varphi(\mathbf{x}(\mathrm{NOT}(a_1 \vee \ldots \vee a_p))) \qquad (4.12)$$

Now because of (4.10) $\varphi(\mathbf{x}(\mathrm{NOT}(a_1 \vee \ldots \vee a_p)))=0$
Therefore, (4.9) holds again. //

Observe that Proposition 4.15 imposes no limitation on the complexity of the statements for which φ will be L-NOR consistent in contrast to Theorem 4.11 for L-NAND consistency.

Proposition 4.16

If the components of a structure φ are L-NOR consistent and $\varphi \in \mathcal{S}_2$ then $\check{C}_{\varphi p}=\emptyset$ for all p and all $\mathcal{A}$.

In words: Structures in $\mathcal{S}_2$ are never contradictory under disjunction of statements.

Proof:

Immediate from Proposition 4.15 and definition of $\check{C}_{\varphi p}$. //

The "opposite" of Proposition 4.16 takes the form of

Proposition 4.17

If the components of a structure are R-NOR consistent and $\check{C}_{\varphi p}=\emptyset$ for all p and all $\mathcal{A}$ then $\varphi \in \mathcal{S}_2$.

Proof:

$\check{C}_{\varphi p}=\emptyset$ implies either of two cases:

Case 1: $\sqcup^{p}{}_{j=1}\varphi(\mathbf{x}(a_j))=0$

In this case $\rho_{Cp}(\varphi;\mathbf{x}(a_1),\dots,\mathbf{x}(a_p)) \equiv$

$$\equiv \sqcup^{p}{}_{j=1}\varphi(\mathbf{x}(a_j))\,\varphi(\mathbf{1}-\sqcup^{p}{}_{j=1}\mathbf{x}(a_j))=0$$

Case 2: $\sqcup^{p}{}_{j=1}\varphi(\mathbf{x}(a_j))=1 \Rightarrow \varphi(\mathbf{x}(\mathrm{NOT}(a_1 \vee \dots \vee a_p)))=0$

where $a_1,\dots,a_p \in \mathcal{A}$. By R-NOR consistency of components

$$1-\sqcup^{p}{}_{j=1}x_i(a_j) \le x_i(\mathrm{NOT}(a_1 \vee \dots \vee a_p))$$

and since φ is coherent

$$\varphi(\mathbf{1}-\sqcup^{p}{}_{j=1}\mathbf{x}_i(a_j)) \le \varphi(\mathbf{x}(\mathrm{NOT}(a_1 \vee \dots \vee a_p)))$$

Hence,

$\varphi(\mathbf{1}-\sqcup^{p}{}_{j=1}\mathbf{x}_i(a_j))=0$

But then in both cases

$\rho_{Cp}(\mathbf{x}(a_1),\ldots,\mathbf{x}(a_p))\equiv\sqcup^{p}{}_{j=1}\varphi(\mathbf{x}(a_j))\varphi(\mathbf{1}-\sqcup^{p}{}_{j=1}\mathbf{x}(a_j))=0$

for all p and a_j.

Therefore, $\varphi\in\mathcal{S}_2$ because of Proposition 1.11. //

Theorem 4.18

If the components of a structure φ are NOR consistent then

(a) $\varphi\in\mathcal{S}_2 \Rightarrow \check{C}_{\varphi p}=\emptyset$ for all p and A.

(b) $\check{C}_{\varphi p}=\emptyset$ for all p and all $A \Rightarrow \varphi\in\mathcal{S}_2$.

Proof:

Immediate from Propositions 4.16, 4.17. //

4.3 Logic Behaviour of $\overline{\mathcal{S}}_p$ Structures

The results of the previous section on $\mathcal{S}_p$ structures can be transformed by symmetrical arguments to results for $\overline{\mathcal{S}}_p$ structures. Now $\mathcal{A}_\varphi$, the passing set, will be replaced by $\overline{\mathcal{A}}_\varphi$, the rejection set. Where AND was mentioned OR will be used instead; and in place of $\mathcal{S}_p$, $\overline{\mathcal{S}}_p$ will appear while the components instead of L-NAND they are going to be R-NOR etc. For this reason we will be rather brief in this section.

Proposition 4.19 (Closure of R-NOR consistency under $\overline{\mathcal{S}}_p$)
If $\varphi \in \overline{\mathcal{S}}_p$ and the components are R-NOR then $\varphi(\mathbf{x}(.))$ is R-NOR for any group of p-1 statements $a_1, \ldots, a_{p-1}$.

Proof:

We want to show that

$$\sqcup^{p-1}{}_{j=1}\varphi(\mathbf{x}(a_j)) \geq 1-\varphi(\mathbf{x}(\mathrm{NOT}(a_1 \vee \ldots \vee a_{p-1}))) \qquad (4.13)$$

If $\sqcup^{p-1}{}_{j=1}\varphi(\mathbf{x}(a_j))=1$ then (4.13) holds.

If $\sqcup^{p-1}{}_{j=1}\varphi(\mathbf{x}(a_j))=0$ then $\varphi(\mathbf{x}(a_j))=0$ for $j=1,\ldots,p-1$

$$(4.14)$$

It follows that there are at most p-1 min cuts that block statements $a_1, \ldots, a_{p-1}$. Suppose now, to the contrary, that there is a min cut that blocks $\mathrm{NOT}(a_1 \vee \ldots \vee a_{p-1})$. Since $\varphi \in \overline{\mathcal{S}}_p$ any p min cuts have at least one common component. Therefore, there is a component, say i, which is common among all p min cuts and for which $x_i(\mathrm{NOT}(a_1 \vee \ldots \vee a_{p-1}))=0$ and $\sqcup^{p-1}{}_{j=1}x_i(a_j)=0$. But x_i is R-NOR thus,

$$0=\sqcup^{p-1}{}_{j=1}x_i(a_j) \geq 1-x_i(\mathrm{NOT}(a_1 \vee \ldots \vee a_{p-1}))=1$$

Contradiction. Therefore, there is no min cut that blocks $NOT(a_1 v \dots v a_{p-1})$. Thus, $\varphi(\mathbf{x}(NOT(a_1 v \dots v a_{p-1})))=1$ and (4.12) holds again. //

This proposition enables us to form $\bar{\mathcal{S}}_p$ structures whose components are $\bar{\mathcal{S}}_p$ structures themselves and be sure that the overall structure will be R-NOR once the components are R-NOR.

Corollary 4.20

If the components are R-NOR then for any $\varphi \in \bar{\mathcal{S}}_p$, $\breve{B}_{\varphi p-1}=\emptyset$

Proof: As for $\mathcal{S}_p$ structures. //

Corollary 4.21

If the components are R-NOR and $\varphi \in \bar{\mathcal{S}}_p$ then $\breve{B}_{\varphi q}=\emptyset$ for all $q \leq p-1$

Proof: Because $\varphi \in \bar{\mathcal{S}}_p \Rightarrow \varphi \in \bar{\mathcal{S}}_{p-1}$. //

Proposition 4.19 and its Corollaries are strengthened in the following Theorem which says that the only structures that are never blocked under disjunction of statements whose OR complexity w.r.t. the rejection set is p-1, are the structures for which any p min cuts have at least one common component and whose components are R-NOR consistent.

Theorem 4.22

Let a structure φ and let A a set of statements so that

$C_{OR}(\overline{\mathcal{A}}_\varphi)=p-1$, then a necessary and sufficient condition for $\check{B}_{\varphi q}=\emptyset$ for all q and A is that $\varphi\in\overline{S}_p$ and its components are R-NOR.

Proof: Symmetric to that for S_p structures. //

Remarks

(1) If φ has q min cuts with $q\leq p-1$ then again the theorem holds.

(2) When p=1 R-NOR is equivalent to R-NOT. Then all the results $\overline{S}_p$ hold for $\overline{S}_2$ and R-NOT consistency.

(3) If A is assumed acyclic under OR logic operations then $\overline{\mathcal{A}}_\varphi$ is acyclic and the problem is trivial.

The theorem requires that $C_{OR}(\overline{\mathcal{A}}_\varphi)=p-1$. One way to achieve this is to limit the number of elements that $\overline{\mathcal{A}}_\varphi$ can contain. We can assure this by restricting the rejection set $\overline{\mathcal{A}}_i$ of each component i. These ideas are studied below.

Proposition 4.23

If $\varphi\in\overline{S}_p$ and $\max_i|\overline{\mathcal{A}}_i|\leq p-1$ then $|\overline{\mathcal{A}}_\varphi|\leq p-1$.

Proof: As for S_p structures. //

Proposition 4.24

$|\overline{\mathcal{A}}_\varphi|\leq p-1 \Rightarrow C_{OR}(\overline{\mathcal{A}}_\varphi)\leq p-1$.

Proof:

For any set of statements B, $|B|\geq C_{OR}(B)$. //

Because of Propositions 4.23, 4.24 we can rephrase Theorem 4.22

and instead of imposing a condition on the complexity of the rejection set of φ we impose a limitation on the number of statements each component is allowed to reject.

Theorem 4.25

Let a structure φ and A a set of statements. Also let $|\overline{A}_i| \leq p-1$ for all i then a necessary and sufficient condition for $\check{B}_{\varphi q}(A) = \emptyset$ for all q and A is that $\varphi \in \overline{\mathcal{S}}_p$ and its components are R-NOR consistent.

Proof: Use Propositions 4.23, 4.24 , Theorem 4.22 and

$$|\overline{\mathcal{A}}_i| \leq p-1 \Rightarrow C_{OR}(\overline{\mathcal{A}}_\varphi) \leq p-1. \qquad //$$

We turn now to $\overline{\mathcal{S}}_2$ structures with R-NAND components.

Proposition 4.26 (Closure of R-NAND under $\overline{\mathcal{S}}_2$)

If the components of a structure φ are R-NAND and $\varphi \in \overline{\mathcal{S}}_2$ then $\varphi(\mathbf{x}(.))$ is R-NAND.

Proof: As for $\mathcal{S}_2$ structures. //

Observe that no limitation is imposed on the complexity of the statements for which φ will be R-NAND in contrast to Theorem 4.22 for R-NOR consistency.

Proposition 4.27

If the components of a structure φ are R-NAND and $\varphi \in \overline{\mathcal{S}}_2$ then $\hat{B}_{\varphi p}(A) = \emptyset$ for all p and A.

In words: Structures in $\overline{\mathcal{S}}_2$ are never blocked under conjunction of statements of any complexity.

Proof: Use Proposition 4.26 and the definition of $\hat{B}_{\varphi p}(A)$. //

The "opposite" of Proposition 4.27 is

Proposition 4.28

If the components of a structure φ are L-NAND and $\hat{B}_{\varphi p}(A)=\emptyset$ for all p and A then $\varphi\in\overline{S}_2$.

Proof: Parallel to that for S_p structures. //

Theorem 4.29

If the components of a structure φ are NAND then $\varphi\in\overline{S}_2 \iff \hat{B}_{\varphi p}(A)=\emptyset$ for all p and A.

Proof: Immediate from Propositions 4.27, 4.28. //

4.4 Logic Behaviour of $\boldsymbol{M}$ Structures

By definition, the self dual structures $\boldsymbol{M}$ are given by

$$\boldsymbol{M}=S_2 \cap \overline{S}_p$$

Now applying the results obtained for S_p and $\overline{S}_p$ structures when p=2, $|\mathcal{A}_\varphi|=p-1=1$, the following are obtained,

Proposition 4.30

A self dual structure φ is NOT consistent for any statement a, if its components are NOT consistent.

Proof:

First observe that L-NAND and L-NOR consistency reduce to L-NOT for statements with complexity 1. Also R-NAND and R-NOR reduce to R-NOT. Now,

(a) $\varphi \in S_2$ is L-NOT for a statement a, if its components are L-NOT. (Recall Propositions 4.8 and 4.15)

(b) $\varphi \in \overline{S}_2$ is R-NOT consistent if its components are R-NOT consistent. (Recall Propositions 4.19 and 4.26)

But $\varphi \in \boldsymbol{M} \equiv S_p \cap \overline{S}_p$ and combining (a), (b) the proof is completed. //

Theorem 4.31

Let $A=\{a, \overline{a}\}$ then,

$[B_\varphi(A)=\emptyset, \quad C_\varphi(A)]=\emptyset, \quad$ for all $A=\{a, \overline{a}\}] \iff$

$[\varphi \in \boldsymbol{M}$ and its components are NOT consistent]

Proof:

If $B_\varphi=\emptyset$ and $C_\varphi=\emptyset$ then $|\mathcal{A}_\varphi| \leq 1$ and $|\overline{\mathcal{A}}_\varphi| \leq 1$.

Use now Theorems 4.11 and 4.22 to show the forward part of the theorem. The opposite is obvious because $\varphi\in\mathcal{S}_2$ and $\varphi\in\overline{\mathcal{S}}_2$. //

Observe that also $T_\varphi=\mathcal{A}_\varphi$ and $F_\varphi=\overline{\mathcal{A}}_\varphi$ and each contains one statement. Theorem 4.31 says that the only structures that are never blocked or contradictory when presented with a pair of statements $\{a, \overline{a}\}$ are the self dual structures whose components are NOT consistent.

A slight variation of the above Theorem can be formulated. Components now are not necessarily NOT consistent but they are constrained to pass one of two statements, that are not necessarily each other's negation, and reject the other. If the structure is self dual then this property transfers to the structure level. To see this, simply, define the one statement to be equivalent to the negation of the other and then the condition that each component passes only one of the two statements is equivalent to NOT consistency and Theorem 4.31 applies.

In Chapter 1 we found that $\mathcal{S}_p\cap\overline{\mathcal{S}}_p=\emptyset$ for $p\geq 3$. Therefore, there is no structure apart from the one component structure that is both never contradictory and never blocked for conjunction or disjunction of statements whose complexity exceeds 2. We have to accept therefore the fact that the most we can get is given by the self dual structures and unfortunately it is good only for sets of statements with two elements only.

The tables below summarize the most important results:

Table 4.1

TYPE OF STRUCTURE

TYPE OF COMPONENT CONSISTENCY

	Coherent	S_p	S_2	$\bar{S}_p$	$\bar{S}_2$
I consistent	I	I	I	I	I
L-NAND		If $C_{AND}(\mathcal{A}_\varphi)$ $\le p-1$ then L-NAND and $\hat{C}_{\varphi q}=\emptyset$ any q.			
R-NAND	Proposition 4.2			R-NAND $\hat{B}_{\varphi q}=\emptyset$ all q.	R-NAND
L-NOR	Proposition 4.4	L-NOR $\check{C}_{\varphi q}=\emptyset$ all q.	L-NOR		
R-NOR				If $C_{OR}(\bar{\mathcal{A}}_\varphi)$ $\le p-1$ then R-NOR and $\check{B}_{\varphi q}=\emptyset$ any q.	
NAND or NOR or [AND, OR, NOT]	I Propos. 4.2, 4.4	L-NOR $\check{C}_{\varphi q}=\emptyset$ all q. If $C_{AND}(\mathcal{A}_\varphi)$ $\le p-1$ then L-NAND and $\hat{C}_{\varphi q}=\emptyset$ any q	L-NOR	R-NAND for all q, $\hat{B}_{\varphi q}=\emptyset$ If $C_{OR}(\bar{\mathcal{A}}_\varphi)$ $\le p-1$ then, R-NOR and $\check{B}_{\varphi q}=\emptyset$ any q	R-NAND

Table 4.2

Summary of Results for Statements with Level of Complexity 1

TYPE OF STRUCTURE

TYPE OF COMPONENT CONSISTENCY	Coherent	$\mathcal{S}_2$	$\overline{\mathcal{S}}_2$	$\mathcal{M}$
L-NOT	Corollary 4.6	L-NOT $C_\varphi = \emptyset$		L-NOT $C_\varphi = \emptyset$
R-NOT	Corollary 4.7		R-NOT $B_\varphi = \emptyset$	R-NOT $B_\varphi = \emptyset$
NOT	Corollaries 4.6, 4.7	L-NOT $C_\varphi = \emptyset$	R-NOT $B_\varphi = \emptyset$	NOT $C_\varphi = \emptyset$, $B_\varphi = \emptyset$ (*)

(*) If $|\mathcal{A}_i| = |\overline{\mathcal{A}}_i| = 1$, $|A| = 2$ then

$|\mathcal{A}_\varphi| = |\overline{\mathcal{A}}_\varphi| = 1$

4.5 Connection with Arrow's Impossibility Theorem

Let a set of statements A that imply a preference structure. Namely, let $A=\{a_1, \ldots, a_m\}$ where the a_j's are of the following form:

a_1: "c_1 R c_2"

a_2: "c_5 R c_3"

.

.

where R symbolizes a preference relation for example the phrase "is preferred or indifferent to".

Components (individuals) are assumed to be able to express their opinions through characteristic functions $x_i(.)$ that do not depend on the set A but only on each alternative alone. They are also supposed to be I consistent and NAND consistent.

We search for a rule F for aggregating individual preferences where

$$F(a_j, A) = \begin{cases} 1 & \text{if society accepts statement a in view of } A \\ 0 & \text{otherwise} \end{cases}$$

We wish the following conditions to be satisfied by the rule F

Condition 1

(a) The number of elements in A is greater than or equal to three.

(b) The rule F is defined for all components in A and any a.

(c) There are at least three components.

Condition 2 (Positive Responsiveness)

If the components' assessments $x_k(a_j)$ on statement a_j is modified to $x_k'(a_j)$ for each individual k so that $x_k'(a_j) \geq x_k(a_j)$ for all k, then the society's new assessment $F'(a_j, A) \geq F(a_j, A)$.

Condition 3 (Independence of Irrelevant Alternatives)

If a statement is added or subtracted from the set of statements A, then F will not change its value as a function of the remaining statements.

Condition 4

The rule F must depend on the components assessments only. And for any statement a_j there are components' assessments $x_1(a_j), \ldots, x_n(a_j)$ so that $F(a_j, A) = 1$.

Condition 5 (No Dictator)

There is no component i so that $F(a_j, A) = x_i(a_j, A)$ for all a_j and for all A.

It is easy to show that Conditions 2, 3, 4 imply that $F(a_j, A)$ can be written as a function $\varphi(x_1(a_j), \ldots, x_n(a_j))$ and further that $\varphi(x_1, \ldots, x_n)$ is a coherent structure. For a proof see Pechlivanides [1975].

Further the rule F is required to obey two Axioms,

Axiom 1

For any two issues c_1, c_2, the rule must decide that either "c_1 R c_2" or that NOT"c_1 R c_2".

In our terminology this requires that the rule F must accept either a or NOTa but not both. This in turn means that F or even better the coherent structure $\varphi(\mathbf{x}(a))$ is NOT consistent.

Axiom 2 (Transitivity)

If the rule F accepts the statement "c_1 R c_2" and the statement "c_2 R c_3" then it must also accept the statement "c_1 R c_3".

This Axiom, in our terminology, requires that the rule F (or the coherent structure $\varphi(\mathbf{x}(a))$ is L-AND consistent.

To see the latter, let

a_1: "c_1 R c_2"

a_2: "c_2 R c_3"

a_3: "c_1 R c_3"

L-AND consistency requires that $\varphi(\mathbf{x}(a_1))\varphi(\mathbf{x}(a_2)) \leq \varphi(\mathbf{x}(a_3))$ which is the same as transitivity. If transitivity was milder expressed as: "if a_1 is accepted and a_2 is accepted then NOTa_3 is rejected", then this is the same as the requirement for L-NAND consistency. However, even if this is the case, we gain nothing because as we show below, L-NAND consistency in the presence of NOT consistency (Axiom 1) for the structure φ, along with I consistency and NAND consistency at the component level lead once more to the requirement for L-AND consistency for φ:

Assuming that components are I consistent so is the structure φ by Proposition 4.1. Hence,

$$\varphi(\mathbf{x}(\overline{a}_3)) \leq \varphi(\mathbf{x}(\mathrm{NOT}(a_1 \wedge a_2))) \quad .$$

Now by L-NAND consistency of φ

$$\varphi(\mathbf{x}(a_1))\varphi(\mathbf{x}(a_2)) \leq 1-\varphi(\mathbf{x}(\mathrm{NOT}(a_1 \wedge a_2)))$$

therefore,

$$\varphi(\mathbf{x}(a_1))\varphi(\mathbf{x}(a_2)) \leq 1-\varphi(\mathbf{x}(\overline{a}_3))$$

But because of NOT consistency (Axiom 1)

$$1-\varphi(\mathbf{x}(\overline{a}_3)) = \varphi(\mathbf{x}(a_3))$$

Therefore,

$$\varphi(\mathbf{x}(a_1))\varphi(\mathbf{x}(a_2)) \leq \varphi(\mathbf{x}(a_3))$$

This says that whenever a_1 passes φ and a_2 passes φ then a_3 passes φ, which is the transitivity requirement and in fact it is the same as L-AND consistency.

We have shown therefore that L-NAND and NOT consistency of φ, along with I consistency and NAND consistency of components imply transitivity of the structure (Axiom 2). The opposite can also be shown by retracing steps:

Transitivity requires that

$$\varphi(\mathbf{x}(a_1))\varphi(\mathbf{x}(a_2)) \leq \varphi(\mathbf{x}(a_3))$$

for all a_3 such that $a_1 \wedge a_2 => a_3$. Therefore,

$$\varphi(\mathbf{x}(a_1))\varphi(\mathbf{x}(a_2)) \leq \varphi(\mathbf{x}(a_1 \wedge a_2))$$

and by NOT consistency of φ (Axiom 1)

$$\varphi(\mathbf{x}(a_1))\varphi(\mathbf{x}(a_2)) \leq 1-\varphi(\mathbf{x}(\mathrm{NOT}(a_1 \wedge a_2)))$$

proving L-NAND consistency of φ.

We expect the Impossibility Theorem to hold in the generalized case where logic statements are used instead of preference rela-

tions since it demonstrates that Conditions 1, 2, 3, 4, 5 and Axioms 1, 2 are incompatible for the particular case when the a_j's represent preference relations.

In fact, in our terminology, we require that the coherent structure φ be:

(a) NOT consistent and therefore belong to the class $\boldsymbol{M}$ of structures.

(b) L-NAND consistent for complexity greater or equal to three (since Condition 1 requires the set of statements to contain at least three statements) and therefore the structure φ must belong to S_3 by Theorem 4.11 .

But then $\varphi \in \boldsymbol{M} \cap S_3$ which is the one component structure because of Theorem 1.7 (see Fig. 1.2). This in turn violates Condition 5 of non-dictator and the Impossibility Theorem holds again.

Still another approach to showing the Impossibility Theorem is to observe that "transitivity" is L-AND consistency. But φ is coherent therefore it obeys I consistency (Proposition 4.1) which is the same as R-AND consistency (Proposition 3.4). Thus we demand that φ be AND consistent. But AND consistency for φ is equivalent to requiring that φ be a series structure. Now adding NOT consistency leads us to the intersection of $\boldsymbol{M}$ with the series structures which yields the one component structure again.

APPENDIX A

Proofs of Properties of Logic Operators that Appear in Chapter 2

O. 1 $a_1 OR a_1 = a_1$

Proof: By A. 1 $a_1 OR a_1 = NOT(\overline{a}_1 AND \overline{a}_1) = NOT\overline{a}_1 = a_1$. //

O. 2 $a_1 OR a_2 = a_2 OR a_1$

Proof: Use A. 2 on AO. 1 //

O. 3 $(a_1 OR a_2) OR a_3 = a_1 OR (a_2 OR a_3)$

Proof: $(a_1 OR a_2) OR a_3 = NOT(NOT(a_1 OR a_2) AND NOT a_3)$

$$= NOT((\overline{a}_1 AND \overline{a}_2) AND \overline{a}_3)$$

$$= NOT(\overline{a}_1 AND (\overline{a}_2 AND \overline{a}_3))$$

$$= NOT(\overline{a}_1 AND (NOT(a_2 OR a_3)))$$

$$= a_1 OR (a_2 OR a_3) \quad //$$

O. 4 $a_1 OR (NOT a_1) = t$

Proof: $a_1 OR \overline{a}_1 = NOT(\overline{a}_1 AND a_1) = NOT f = t$ //

O. 5 $a_1 OR t = t$

Proof: $a_1 OR t = a_1 OR (a_1 OR (NOT a_1))$

$$= (a_1 OR a_1) OR (NOT a_1)$$

$$= a_1 OR \overline{a}_1 = t \quad //$$

AO. 5 $a_1 AND (a_1 OR a_2) = a_1$

Proof: By AO. 4 $a_1 AND (a_2 OR a_3) = (a_1 AND a_2) OR (a_1 AND a_3)$

Let $a_1 = a_3$ then

$a_1 AND (a_2 OR a_1) = (a_1 AND a_2) OR a_1$

But using AO. 3

$$= a_1 \quad //$$

AO. 6 $a_1 OR (a_2 AND a_3) = (a_1 OR a_2) AND (a_1 OR a_3)$

Proof: $(a_1 \mathrm{OR} a_2)\mathrm{AND}(a_1 \mathrm{OR} a_3)=[(a_1 \mathrm{OR} a_2)\mathrm{AND} a_1]\mathrm{OR}[(a_1 \mathrm{OR} a_2)\mathrm{AND} a_3]=$

$[(a_1 \mathrm{AND} a_1)\mathrm{OR}(a_1 \mathrm{AND} a_2)]\mathrm{OR}[(a_1 \mathrm{AND} a_3)\mathrm{OR}(a_2 \mathrm{AND} a_3)]=$

$=[a_1 \mathrm{OR}(a_1 \mathrm{AND} a_2)]\mathrm{OR}[(a_1 \mathrm{AND} a_3)\mathrm{OR}(a_2 \mathrm{AND} a_3)]=$

$=a_1 \mathrm{OR}[a_2 \mathrm{AND} a_3]$ //

I.3 $(a_1=>a_2)<=>(\overline{a}_2=>\overline{a}_1)$

Proof: $(a_1=>a_2)=(\overline{a}_1 \vee a_2)=(a_2 \vee \overline{a}_1)=(\mathrm{NOT}(\overline{a}_2)\vee \overline{a}_1)=(\overline{a}_2=>\overline{a}_1)$

I.4 $(a_1=>a_2)=>((a_1 \wedge a_2)<=>a_1)$

Proof:

(i) $a_1 \wedge a_2=>a_1$: $\mathrm{NOT}(a_1 \wedge a_2)\vee a_1=\overline{a}_1 \mathrm{OR} \overline{a}_2 \mathrm{OR} a_1=\overline{a}_2 \mathrm{OR} t=t$

(ii) $a_1=>a_1 \wedge a_2$:

$\overline{a}_1 \vee (a_1 \wedge a_2)=(\overline{a}_1 \vee a_1)\wedge(\overline{a}_1 \vee a_2)=t\wedge(\overline{a}_1 \vee a_2)=(a_1=>a_2)$ which holds whenever the LHS of I.4 holds. //

I.5 $(a_1=>a_2)=>((a_1 \vee a_2)<=>a_2)$

Proof: Similar to I.4 //

I.6 $((a_1=>a_2)\mathrm{AND}(a_2=>a_3))=>(a_1=>a_3)$

Proof: We have to show that

$[\mathrm{NOT}((\overline{a}_1 \vee a_2)\mathrm{AND}(\overline{a}_2 \wedge a_3))]\mathrm{OR}(\overline{a}_1 \vee a_3)=t$

$[(\mathrm{NOT}(\overline{a}_1 \vee a_2))\vee(\mathrm{NOT}(\overline{a}_2 \vee a_3))]\vee(\overline{a}_1 \vee a_3)=$

$=[(a_1 \wedge \overline{a}_2)\vee(a_2 \wedge \overline{a}_3)]\vee(\overline{a}_1 \vee a_3)=$

$=[(a_1 \vee a_2)\wedge(a_1 \vee \overline{a}_3)\wedge(\overline{a}_2 \vee a_2)\wedge(\overline{a}_2 \vee \overline{a}_3)]\vee(\overline{a}_1 \vee a_3)=$

$=[(a_1 \vee a_2)\vee(\overline{a}_1 \vee a_3)]\wedge[(a_1 \vee \overline{a}_3)\vee(\overline{a}_1 \vee a_3)]\wedge[(\overline{a}_2 \vee \overline{a}_3)\vee(\overline{a}_1 \vee a_3)$
$]=$

$=(t \vee a_2 \vee a_3)\wedge(t \vee t)\wedge(t \vee \overline{a}_2 \vee \overline{a}_1)=t$ //

I.7 $(a=>b)=>$(there is c so that $b \wedge c=a$)

Proof: Take $c=(a \vee \overline{b})\wedge(b \vee \overline{a})$

(Note that $c=\mathrm{NOT}(a \mathrm{XOR} b)$)

$b \wedge c=b \wedge[(a \vee \overline{b})\wedge(b \vee \overline{a})]=b \wedge(a \vee \overline{b})\wedge(b \vee \overline{a})$

but $b \vee \overline{a}$ is assumed true hence,

$= b \wedge (a \vee \overline{b}) = (b \wedge a) \vee (b \wedge \overline{b}) = b \wedge a$

and because of I.4

$= a$ //

I.8 $(a => b) => $ (there is c so that $a \vee c = b$)

Proof: Take $c = (\overline{a} \vee \overline{b}) \wedge (a \vee b) = a \mathrm{XOR} b$

Then show that $a \vee c = b$ as in I.7 using now I.5 instead of I.4 //

BOOK 2

STRUCTURAL PROPERTIES

Chapter 5

OPERATIONS ON STRUCTURES

5.1 The Path and the Cut Matrix of a Coherent Structure

Let the min paths of the coherent structure $\varphi(x_1,\ldots,x_n)$ be given by the sets of components $P_1, P_2, \ldots, P_m$. Then define the *min path matrix* of a coherent structure φ, $\mathbf{P}_\varphi$ as the mxn matrix whose elements are $[\pi_{ij}]$:

$$\pi_{ij} = \begin{cases} 1 & \text{if component } j \text{ appears in min path } i \\ 0 & \text{otherwise} \end{cases}$$

Example 5.1

Let φ have the following min paths

$P_1 = \{1, 2\}$, $P_2 = \{2, 3, 4\}$, $P_3 = \{3, 5\}$

Then the min path matrix $\mathbf{P}_\varphi$ is given by,

$$\mathbf{P}_\varphi = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Similarly, the *min cut matrix* of a structure $\varphi(x_1,\ldots,x_n)$ denoted $\mathbf{C}_\varphi$ is defined as the mxn matrix whose elements are $[\gamma_{ij}]$:

$$\gamma_{ij} = \begin{cases} 1 & \text{if component } j \text{ appears in min cut } i \\ 0 & \text{otherwise} \end{cases}$$

Observe that $\mathbf{P}_\varphi = \mathbf{C}_\varphi D$ and $\mathbf{C}_\varphi = \mathbf{P}_\varphi D$ since the min paths of the dual structure are the min cuts of the original and

vice versa. It follows that for self dual structures ($\varphi = \varphi^D$)

$$\mathbf{P}_\varphi = \mathbf{P}_{\varphi^D} = \mathbf{C}_\varphi = \mathbf{C}_{\varphi^D}$$

Also note that no row is subset of another row since each row represents a min path (cut) and this is not allowed by their definition (they would be redundant).

5.2 Contraction of Coherent Structures

Consider a coherent structure where two components decide to vote always in the same way. Then the structure behaves as if it had one less component (since the second component can be replaced by a repetition of the first in each min path it appears). In fact, we can *contract* more than two components in groups and in this way reach new structures that keep essential properties of the original structures.
In what follows we will first define *contraction* and then study the effects of contraction on structures in $\mathcal{S}_p$, $\bar{\mathcal{S}}_p$, $\mathcal{M}$.

Let a structure $\varphi_n(\mathbf{x})$ where as usual $\mathbf{x}=x_1,\ldots,x_n$ and the subscript n of φ_n reminds that φ_n is a n component structure.
Let $N=\{1,\ldots,n\}$, $G\subseteq N$ and $|G|=g$

Definition

The contraction of a n component structure $\varphi_n(x_1,\ldots,x_n)$ with respect to the group of components G ($G\subseteq N$), symbolized as $K(G|\varphi_n)$ is another structure φ'_{n-g+1} so that,

$\varphi_n(x_1,\ldots,x_n)=\varphi'_{n-g+1}(x_{i(1)},\ldots,x_{i(l)},x_\Gamma)$

$(=K(G|\varphi_n))$

for all $x_1,\ldots,x_n$ such that $\{i_1,i_2,\ldots,i_l\}=N-G$ and

$x_j=x_\Gamma$ for all $j\in G$.

Note that all x_j with $j\in G$ are replaced by x_Γ .

Let now G_1, $G_2 \in N$, $G_1 \cap G_2 = \emptyset$, $G_1 \neq \emptyset$, $G_2 \neq \emptyset$, $G_1 = \{k_1, \ldots, k_{g(1)}\}$, $G_2 = \{j_1, \ldots j_{g(2)}\}$, $N - G_1 - G_2 = \{i_1, \ldots, i_l\}$.

Denote $K(G_2, G_1 | \varphi) = K(G_2 | K(G_1 | \varphi))$ then the order of contraction is not important.

Proposition 5.2

$K(G_2, G_1 | \varphi) = K(G_1, G_2 | \varphi)$

Proof:

$$K(G_2, G_1 | \varphi_n(x_{i(1)}, \ldots, x_{i(l)}; x_{k(1)}, \ldots, x_{k(g1)}; x_{j(1)}, \ldots \\ \ldots, x_{j(g2)})) =$$

$$= K(G_2 | K(G_1 | \varphi)) =$$

$$= K(G_2 | \varphi'(x_{i(1)}, \ldots, x_{i(l)}; x_{\Gamma 1}; x_{j(1)}, \ldots, x_{j(g2)})) =$$

$$= \varphi''(x_{i(1)}, \ldots, x_{i(l)}; x_{\Gamma 1}; x_{\Gamma 2})$$

where $\varphi' = K(G_1 | \varphi)$

Therefore, $\varphi''_{n-g1-g2+2} = \varphi_n$

for all $x_{i(1)}, \ldots, x_{i(l)}; x_{k(1)}, \ldots, x_{k(g1)}; x_{j(1)}, \ldots, x_{j(g2)}$ so that $\{i_1, \ldots, i_l\} = N - G_1 - G_2$ and $x_k = x_{\Gamma 1}$ for all $k \in G_1$ and $x_j = x_{\Gamma 2}$ for all $j \in G_2$ (5.1)

Applying now the contraction in reverse order,

$K(G_1, G_2 | \varphi_n) = K(G_1 | K(G_2 | \varphi_n))$

we obtain say φ''' which again satisfies the same property (5.1) as φ''. Hence $\varphi''' = \varphi''$ as we wanted to show.

The generalization to multiple contractions can now be defined. Let a partition of N to non overlapping, non empty groups of components $G_1, G_2, \ldots, G_g \subseteq N$ and define,

$K(G_1, \ldots, G_g | \varphi) \equiv$

$$\equiv K(G_1 | K(G_2 | K(\ldots K(G|\varphi)\ldots)))$$

and again the order of contractions is not important.

Finally, it can be shown that successive contractions that may overlap are always equivalent to a contraction of some non overlapping groups of components of the original structure:

Let us examine $K(G_2|K(G_1|\varphi_n))$ where $G_1=\{k_1, \ldots, k_{g1}\}$, $g_1<n$ and let $x_{\Gamma 1}$ be the common component that replaces $x_{k(1)}, \ldots, x_{k(g1)}$ which are contracted in $K(G_1|\varphi_n)$.

Suppose now that $G_2=\{\Gamma_1, j_1, \ldots, j_{g2-1}\}$, $|G_2|=g_2$

Then $K(G_2|K(G_1|\varphi_n))=K(G_2 \cup G_1-\{\Gamma_1\}|\varphi_n)$

This says that instead of making two successive overlapping contractions we may contract the initial structure with respect to $G_1 \cup G_2-\{\Gamma_1\}$. To prove this we start with the definition of contraction,

$$K(G_2|K(G_1|\varphi_n))=K(G_2|\varphi'_{n-g1+1}(x_{i(1)}, \ldots, x_{i(1)}; x_{\Gamma 1}; x_{j(1)}, \ldots, x_{j(g2-1)};))$$

where

$\varphi'_{n-g1+1}=K(G_1|\varphi_n)$

and $\{i_1, \ldots, i_1\}=N-G_1-(G_2-\{x_{\Gamma 1}\})$

Then

$$K(G_2|K(G_1|\varphi_n))=\varphi''_{n-g1-g2+2}(x_{i(1)}, \ldots, x_{i(1)}; x_{\Gamma 2})$$

where $x_{\Gamma 2}$ is the common component that replaces the components in G_2 after $K(G_2|\varphi')$ is applied.

By the definition of contraction

$\varphi''_{n-g1-g2+2}(x_{i(1)}, \ldots, x_{i(1)}; x_{\Gamma 2})=$

$\varphi(x_{i(1)},\ldots,x_{i(l)};x_{K(1)},\ldots,x_{K(g1)};x_{j(1)},\ldots,x_{j(g2-1)}$

for all $x_{i(1)},\ldots,x_{i(l)};x_{K(1)},\ldots,x_{K(g1)};x_{j(1)},\ldots,x_{j(g2-1)}$ such that

$x_{K(1)}=\ldots=x_{K(g1)}=x_{\Gamma 1}=x_{j(1)}=\ldots=x_{j(g2-1)}=x_{\Gamma 2}$

But this is equivalent to the definition of

$K(G_1\cup G_2-\{\Gamma_1\}|\varphi_n)$ as we wanted to show.

In general we let

$G_1,G_2,\ldots,G_{K-1}\subseteq N,\quad G_i\cap G_j=\emptyset$ for all

$i,j\in\{1,\ldots,K-1\}$ Then,

$K(G_K|K(G_1,G_2,\ldots,G_{K-1}|\varphi))=K(G_1,G_2,\ldots$

$\ldots,G_l,G_{l+1}\cup\ldots\cup G_{K-1}\cup G_K-\{\Gamma_{l+1},\Gamma_{l+2},\ldots$

$\ldots,\Gamma_{K-1}\}|\varphi)$

where

$\Gamma_i\notin G_K$ for $i=1,\ldots,l$

$\Gamma_i\in G_K$ for $i=l+1,\ldots,K-1$

and

$G_K\subset N-G_1-G_2-\ldots-G_{K-1}\cup\{\Gamma_1,\ldots,\Gamma_{K-1}\}.$

Example 5.3

Let the structure φ with min paths,

1, 2, 3, 4

4, 5, 6, 7

1, 2, 5, 7

3, 4, 5

Let $G_1=\{1,2\}$ and let Γ_1 be the new component replacing components 1 and 2. Then the contracted structure $K(G_1|\varphi)$ will have the following min paths,

Γ_1, 3, 4

4, 5, 6, 7

Γ_1, 5, 7

3, 4, 5

Now let $G_2=\{\Gamma_1, 3\}$ and let Γ_2 be the new component replacing Γ_1 and 3. Then the structure $K(G_2|K(G_1|\varphi))$ will have the following min paths,

Γ_2, 4

4, 5, 6, 7

Γ_2, 5, 7

Γ_2, 4, 5

The last path is redundant in the presence of the first path and should be discarded. Also observe that

$$K(G_2|K(G_1|\varphi))=K(G_1 \cup G_2-\{\Gamma_1\}|\varphi)=$$
$$=K(\{1,2,3\}|\varphi)$$

as expected.

Discussion

Contraction is not just a mathematical concept, it has real life meaning in decision structures and it appears as the formation of political parties or when people are represented by proxies or other similar cases of formation of groups which present their opinion as a block, although they may occupy different positions within the decision structure.

Example 5.4 (Effect of contraction on $\mathbf{P}_\varphi$ and $\mathbf{C}_\varphi$)

The operation of contraction of components j and k to a new component l changes the min path matrix $\mathbf{P}_\varphi$ of a structure φ

by:

(a) Merging columns j and k to a new column l whose elements π_{il} are given by:

$$\pi_{il}=\max(\pi_{ij},\pi_{ik}) \quad i=1,\ldots,n$$

where $\mathbf{P}_\varphi$ is mxn.

(b) Omitting those rows (if any) that after the merging in (a) become redundant; and thus cease to represent min paths.

The operation is exactly the same for $\mathbf{C}_\varphi$.

Let for example a structure φ whose $\mathbf{P}_\varphi$ is given by:

$$\mathbf{P}_\varphi = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Let us apply contraction $G=\{3,5\}$ and call the new component that replaces 3 and 5 by 3′. The new structure φ' will have a min path matrix $\mathbf{P}_\varphi'$:

$$\mathbf{P}_\varphi' = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3' & 4 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Observe that $\mathbf{P}_\varphi'$ is indeed a min path matrix and that no row is redundant.

If, instead, we apply the contraction $G=\{2,6\}$ on φ and call the new component that replaces them 2′, $\mathbf{P}_\varphi$ will change (after merging columns 2, 6) to:

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2' & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

The third row is redundant in the presence of row 2 and must be omitted if the matrix is to represent a min path matrix:

$$\begin{array}{c} \\ 1 \\ 2 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2' & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

The closedness of the operation of contraction for structures in $\mathcal{S}_p$ or $\overline{\mathcal{S}}_p$ or $\mathcal{M}$ is straightforward. If $\varphi \in \mathcal{S}_p$ and a contraction is applied to a group of its components, this will not change the fact that any p min paths of φ, and, hence, of the new structure, will continue to have at least one common component. In fact, we expect that contraction will probably lead to φ' so that $\varphi' \in \mathcal{S}_{p'}$ with $p' \geq p$. Similarly with $\overline{\mathcal{S}}_p$.

If $\varphi \in \mathcal{M}$ any contraction will lead to φ' with $\varphi' \in \mathcal{M}$ since $\varphi \in \mathcal{S}_2$ and $\varphi \in \overline{\mathcal{S}}_2 \Rightarrow$

$\Rightarrow \varphi' \in \mathcal{S}_2$ and $\varphi' \in \overline{\mathcal{S}}_2 \Rightarrow \varphi' \in \mathcal{M}$.

5.3 Contractions on Symmetric Structures

Let φ be a symmetric (k out of n) structure whose components $\{1,\ldots,n\}$, have characteristic functions $x_1,\ldots,x_n$.
Suppose that we contract components in the group $G=\{i_1,\ldots,i_g\}$ and because of symmetry of components we may write without loss of generality $G=\{1,\ldots,g\}$. After contraction we will be left with a structure of n-G+1 components ($G\equiv|G|$) containing the following min paths:

(a) Min paths of φ that do not contain any of the components of G. There are $\binom{n-G}{k}$ such paths with k components each. If, however, $n-G<k$ then no min path will exist in φ that does not contain a component of G.

(b) Min paths of φ that contain all of the components in G (assuming that $G\le k$). There are $\binom{n-G}{k-G}$ such paths with $k-G+1$ components each. If, however, $G\ge k+1$, then there will be only one min path containing one component apart from those in case (a).

All other min paths of φ containing a subset of G plus some other components not in G will be redundant in the presence of the paths in (b).

To generalize, let φ (k out of n) symmetric structure be subjected to contraction $K(G_1,G_2,\ldots,G_g|\varphi)$. The structure will be contracted to a $n-\Sigma^{g}_{i=1}G_i+g$ component structure ($G_i\equiv|G_i|$) containing the following min paths:

(0) All min paths of φ that contain no component from any of

the G_i's. these paths will exist if $n-\Sigma^{g}_{i=1}G_i \geq k$. There will be

$$\binom{n-\Sigma G_i}{k}$$

such min paths.

(1) All min paths of φ that contain:

* All of G_1 (when $G_1 \leq k$) and no component from $\cup^{g}_{i=1}G_i - G_1$. There will be

$$\binom{n-\Sigma G_i}{k-G_1}$$

such paths.

Or all of G_1 (when $G_1 > k$) and there is only one such path and has only one component.

.

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* All of G_g (when $G_g \leq k$) and no component from $\cup^{g}_{i=1}G_i - G_g$. There will be

$$\binom{n-\Sigma G_i}{k-G_g}$$

such paths.

Or all of G_g (when $G_g > k$) and there is only one such path.

(2) (a) For each pair (G_i, G_j), all the min paths of φ that contain all components of $G_i \cup G_j$ and none from G_l for all $l \neq i, j$. There are

$$\binom{n-\Sigma G_i}{k-G_i-G_j}$$

such min paths for each pair (i, j) when $k \geq G_i+G_j$.

(b) For each pair (G_i, G_j), one min path with two components if $k \leq G_i+G_j$ but $k>G_i$, $k>G_j$.

(3) (a) For each triplet (G_i, G_j, G_m), all the min paths of φ that contain all components of $G_i \cup G_j \cup G_m$ and no other component from G_r, for all $r \in \{1, \ldots, g\} - \{i, j, m\}$. There are

$$\binom{n-\Sigma G_i}{k-G_i-G_j-G_m}$$

such min paths for each triplet (i, j, m) when $k \geq G_i+G_j+G_m$.

(b) For each triplet (G_i, G_j, G_m) where $k \leq G_i+G_j+G_m$ but $k>\max(G_i+G_j, G_i+G_m, G_j+G_m)$, one min path with three components.

.

.

.

(g) (a) All min paths of φ that contain all of the components in $G_1 \cup \ldots \cup G_g$. There are

$$\binom{n-\Sigma G_i}{k-\Sigma G_i}$$

such min paths when $k \geq \Sigma G_i$; otherwise,

(b) If $k<\Sigma G_i$ but $k>$(the number of components of any g-1 of the G_i's) then it contains one min path with g components.

5.4 Omission of Min Paths (OP) or Min Cuts (OC)

By omitting a row from $\mathbf{P}_\varphi$ when $\varphi \in \mathcal{S}_p$, a new matrix $\mathbf{P}_{\varphi'}$ is reached where $\varphi' \in \mathcal{S}_p$ since the remaining rows will continue to represent min paths and will continue to keep the property that any p of them have at least one common component. The same argument holds for $\mathbf{C}_\varphi$ where $\varphi \in \overline{\mathcal{S}}_p$. Namely, the omission of a row from $\mathbf{C}_\varphi$ leads to the min cut matrix of another structure φ', $\mathbf{C}_{\varphi'}$. Again $\varphi' \in \overline{\mathcal{S}}_p$. These arguments do not hold if a min cut is omitted from $\varphi \in \mathcal{S}_p$ or a min path is omitted from $\varphi \in \overline{\mathcal{S}}_p$.

However, for structures in $\boldsymbol{M}$ the situation is different. If a row is omitted from $\mathbf{P}_\varphi$ when $\varphi \in \boldsymbol{M}$ then the resulting structure $\varphi' \notin \boldsymbol{M}$ but $\varphi' \in \mathcal{S}_2$. Similarly, once a row is omitted from $\mathbf{C}_\varphi$ when $\varphi \in \boldsymbol{M}$ then the new structure $\varphi' \notin \boldsymbol{M}$ but $\varphi' \in \overline{\mathcal{S}}_2$. To support this argument it suffices to prove the following,

Proposition 5.5

Let the structures φ_1, φ_2 both belong to the class $\boldsymbol{M}$ of self dual structures. Let also

$\mathcal{P}_1$: the set of paths of φ_1

$\mathcal{P}_2$: the set of min paths of φ_2

Then neither of $\mathcal{P}_1$ or $\mathcal{P}_2$ is a proper subset of the other.

Proof:

Suppose to the contrary that the min paths of φ_2 consist of all the min paths of φ_1 plus some other min paths in the set Q so that, $\mathcal{P}_2=\mathcal{P}_1 \cup Q$, $\mathcal{P}_1 \cap Q=\emptyset$, $Q\neq\emptyset$.

Take a path in Q and call it q. Since $\varphi_2 \in \boldsymbol{M}$ and q is a min path of φ_2, it is also a min cut of φ_2. Since q is a min cut of the paths in $\mathcal{P}_2$ it is also a cut (not necessarily min cut) of $\mathcal{P}_1$, since $\mathcal{P}_1 \subset \mathcal{P}_2$.

We will show that q is also a min cut of $\mathcal{P}_1$ which will lead us to contradiction thus proving the proposition. Let therefore, to the contrary, that q is not also a min cut of $\mathcal{P}_1$. This means that some min cut of $\mathcal{P}_1$, say r, is a proper subset of q. But $\varphi_1 \in \boldsymbol{M}$, thus, r is a min path of φ_1 and therefore $r \in \mathcal{P}_1$. It follows that since $\mathcal{P}_2 \supset \mathcal{P}_1$, $r \in \mathcal{P}_2$. But now φ_2 has two min paths r and q and r is a proper subset of q. This is impossible by the definition of min path and therefore our assumption that q was not a *min* cut is wrong. Thus q is a *min* cut of φ_1.

Now since $\varphi_1 \in \boldsymbol{M}$ and q is a min cut of φ_1, then q is min path of φ_1 and therefore $q \in \mathcal{P}_1$. Contradiction, since $q \in Q$. Therefore, $\mathcal{P}_1$ is not a proper subset of $\mathcal{P}_2$ and vice versa. //

Certainly Proposition 5.5 holds for cuts as well.

Another property of structures in $\boldsymbol{M}$ is revealed by the following Proposition 5.6. But first some notation is needed,

P_{1i}: the ith min path of φ_1

P_{2i}: the ith min path of φ_2

also let

$\mathcal{P}_1=\{P_{11},\ldots,P_{1m}\}$, the set of min paths of φ_1

$\mathcal{P}_2=\{P_{21},\ldots,P_{2s}\}$, the set of min paths of φ_2

The proposition below roughly says that if a structure has min paths which are supersets of those of a structure in $\boldsymbol{M}$, then this structure cannot also belong to $\boldsymbol{M}$.

Proposition 5.6

Let φ_1, φ_2 two coherent structures, and let for each $i\in\{1,\ldots,s\}$ be a $k\in\{1,\ldots,m\}$ so that $P_{2i}\supseteq P_{1k}$ and let there be an $i'\in\{1,\ldots,s\}$ and a $k'\in\{1,\ldots,m\}$ so that $P_{2i'}\supset P_{1k'}$, then

$$\varphi_1\in\boldsymbol{M} \Rightarrow \varphi_2\notin\boldsymbol{M}$$

Proof:

Since $\varphi_1\in\boldsymbol{M}$, the set of min paths $\mathcal{P}_1$ is also the set of min cuts of the min paths in $\mathcal{P}_1$. Look now at $\mathcal{P}_2$. Since each $P_{2i}\supseteq P_{1k}$, it follows that the set of min cuts of $\mathcal{P}_1$ is also a set of cuts for $\mathcal{P}_2$. But then we have identified a cut of $\mathcal{P}_2$, namely, $P_{1k'}\in\mathcal{P}_1$ which is a strict subset of $P_{2i'}\in\mathcal{P}_2$ (by assumption). Therefore, $P_{2i'}$ is not a min cut of $\mathcal{P}_2$. Therefore, $\varphi_2\notin\boldsymbol{M}$. //

Discussion

Omission of min cuts or min paths find their place in real life decision structures when certain coalitions of components are not allowed either as passing coalitions (paths) or blocking coali-

tions (cuts). Such a situation appears when we have "disagreement" groups of components. Disagreement groups can be defined in many different ways according to the real life case they depict. One such simple case is the *simple disagreement groups* that are defined in two ways:

(a) *Simple passing disagreement*: A group R of components is said to be a group of *passing disagreement* if for each $a \in A$, whenever there is an $i \in R$ with $x_i(a)=1$ then $x_j(a)=0$ for all $j \in R-\{i\}$.

(b) *Simple blocking disagreement*: A group R of components is said to be a group of blocking disagreement if for each $a \in A$, whenever there is an $i \in R$ so that $x_i(a)=0$ then $x_j(a)=1$ for all $j \in R-\{i\}$.

It is obvious that components with simple passing disagreement will render a path ineffective if they are both on that path. This implies the omission of that path. Similarly for simple blocking disagreement and cuts.

More complicated rules for disagreement may exist. For example, component i and j are in passing disagreement but not when in the same path with component k. In this case paths that contain both i and j but not k must be omitted.

The order of performing the operation of contraction, K, and OP is important. Let OP be defined by some disagreement rule among components. If OP is performed first on φ, then some path that would not be redundant if K was instead performed first, may be omitted while others that would be redundant in its presence after K, now that it has disappeared, they are kept.

If in particular, OP is the result of only simple disagreement

rules then the order of performing OP (OC) and K is not important because there is no danger that OP (OC), when performed first will eliminate a path that would not be redundant if K was first performed instead. But the ideas of performing OP (OC) and K will be further studied in Chapter 6.

The effects of contraction K and omission of paths OP or cuts OC can be shown in the following figures

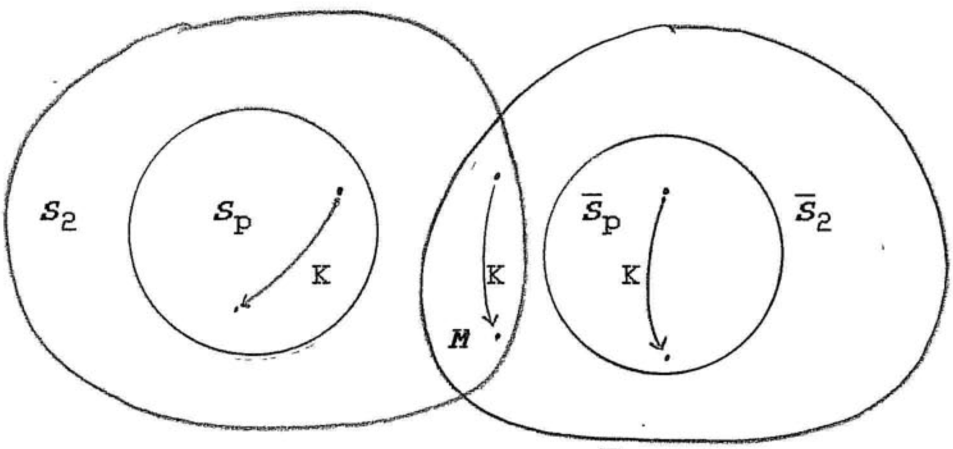

Figure 5.1 Closedness of S_p, $\bar{S}_p$, M under contraction K

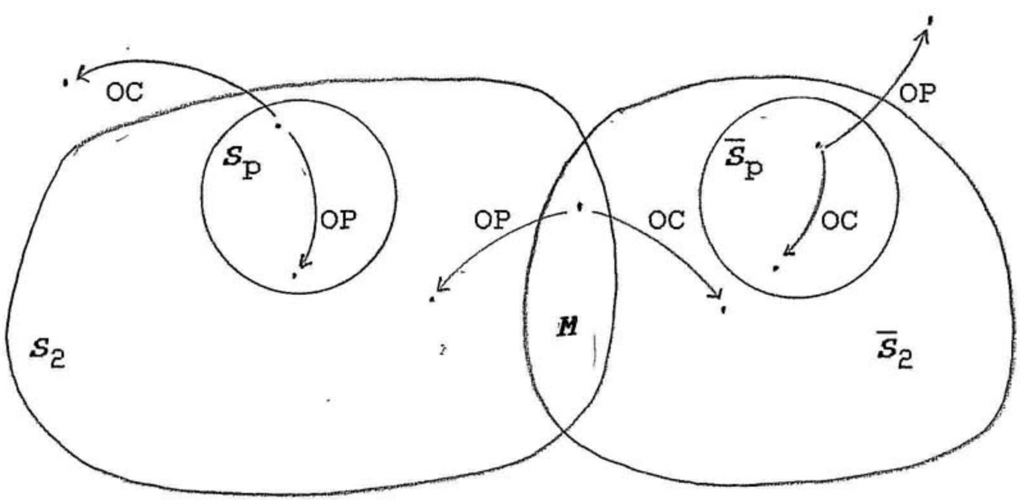

Figure 5.2 Closedness of S_p ($\bar{S}_p$) under OP (OC).

Non closedness of M under OP or OC.

Example 5.7

A committee is formed by the mayor of a town to study and propose alternative ways for waste collection to be presented to the city council for final decision.

The committee consists of 10 members and a proposal will be accepted for submission to the council if it is supported by any three of the members of the committee (3 out of 10 symmetric structure). The members are labeled by a characteristic, say Male or Female, and suppose there are 5 males and 5 females. For reasons paradoxical to us, the council has ruled that it will not consider proposals supported only by females or only by males.

The structure then is a 3/10 structure where OP has been applied on min paths with three males or three females.

If further there is a schism among males and females regarding waste collection, leading to the formation of two opposing parties so that the members of each of them vote homogeneously, then we have a contraction of the 3/10 structure, that was subjected to OP as described above, to a series structure of two components: M, F. (M=Male, F=Female).

Chapter 6

THE DERIVABILITY OF STRUCTURES FROM SYMMETRIC STRUCTURES

By the operation of contraction, K, and omission of paths, OP, or cuts, OC, we are able to reach other structures. In particular, operations of K, OP, OC on symmetric structures will lead to other structures in S_2, $\bar{S}_2$, M, C.

Now we want to study the opposite problem: Given some coherent structure φ, is there always some symmetric structure(s) from which we can derive it, and by what sequence of operations of K, OP, OC?

In fact it is shown that any coherent structure is derivable from

a symmetric structure (call it mother structure) by the sequence of operations of (OP and K) or (OC and K).

Further conditions for deriving a structure from a symmetric structure by the sequence of operations of (K and OP) or (K and OC) are found.

6.1 Derivation of Structures from Symmetric Structures by (K and OP) or by (K and OC)

First some definitions and notation.

Symmetric structures will often be referred to as "k/n" or "k out of n" or "$\varphi_{k/n}$".

The missing element matrix of $\mathbf{P}_\varphi$ denoted $\mathbf{P}_\varphi{}'$ is defined as the matrix that is derived from $\mathbf{P}_\varphi$ by the following operations: Each row j of $\mathbf{P}_\varphi$ that contains components $\{j_1, \ldots, j_J\}$ is replaced by the J different rows that can be formed by omitting one element from the set $\{j_1, \ldots, j_J\}$ each time (see example below).

The *missing element matrix* of $\mathbf{C}_\varphi$ is similarly defined.

Example 6.1

Let

$$\mathbf{P}_\varphi = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

then

$$
\mathbf{P}_\varphi{}' = \begin{array}{c|cccc}
 & 1 & 2 & 3 & 4 \\
\hline
1a & 1 & 1 & 0 & 0 \\
1b & 1 & 0 & 1 & 0 \\
1c & 0 & 1 & 1 & 0 \\
2a & 0 & 1 & 0 & 0 \\
2b & 0 & 0 & 0 & 1 \\
3a & 1 & 0 & 1 & 0 \\
3b & 1 & 0 & 0 & 1 \\
3c & 0 & 0 & 1 & 1
\end{array}
$$

Let us define now the following problems,

Problem *P. 1* (φ)

$\mathbf{P}_\varphi \mathbf{y} \geq z\mathbf{e}$

$\mathbf{P}_\varphi{}' \mathbf{y} \leq (z-1)\mathbf{e}$

$z, \mathbf{y}$ integer positive

Problem *C. 1* (φ)

$\mathbf{C}_\varphi \mathbf{x} \geq w\mathbf{e}$

$\mathbf{C}_\varphi{}' \mathbf{x} \leq (w-1)\mathbf{e}$

$w, \mathbf{x}$ positive integer

where

$\mathbf{P}_\varphi$ is the min path matrix of φ, (pxn)

$\mathbf{C}_\varphi$ is the min cut matrix of φ, (cxn)

$\mathbf{P}_\varphi{}'$ is the missing element matrix of $\mathbf{P}_\varphi$

$\mathbf{C}_\varphi{}'$ is the missing element matrix of $\mathbf{C}_\varphi$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}$$

z, w are scalars and **e** is a vector of ones of appropriate dimension.

The requirement that **y**, **x** are integer positive is that all their elements are integer and positive.

In this chapter we will often use the notation k/N for symmetric structures (k out of N) instead of the usual k/n because n will be used here to denote the number of components of some structure φ that is associated with the min path matrix $\mathbf{P}_\varphi$. When there is no danger of confusion k/n is used again. Also Σy_i will often be used instead of $\Sigma^n{}_{i=1}y_i$ and Σw_i instead of $\Sigma^n{}_{i=1}w_i$.

Theorem 6.2

If Problem *P.1*(φ) has a solution z, **y**, then φ can be reached from a symmetric structure (namely the z out of Σy_i structure) by contraction and then by omission of paths (K, OP)

Proof:

Consider the structure φ_o which is z out of Σy_i. Pick non overlapping groups of components so that,

the first group G_1 contains y_1 components

the second group G_2 contains y_2 components

.   .

.   .

the nth group G_n contains y_n components

Perform the contraction $K(G_1,...,G_n|\varphi_o)=\varphi_1$. We will show that the set of min paths of φ_1 contains all the min paths of φ. To prove this take any min path of φ. Let this min path be composed of components $\{i_1,...,i_l\}$. Since **y** is a solution to *P.1*(φ) we know that it satisfies,

$$\Sigma^l_{j=1} y_{i(j)} \geq z \quad \text{and}$$

$$\Sigma^l_{j=1} y_{i(j)} - y_{i(k)} \leq z-1 \quad \text{for any } k \in \{1,...,l\}$$

Now we have to show that the above min path of φ is also a min path of φ_1. Pick that min path from φ_o structure, that contains the groups $G_{i(1)},...,G_{i(l-1)}$ and a portion of the components in the group $G_{i(l)}$. Such a min path exists in φ_o since it is symmetric and because it satisfies,

$$\Sigma^l_{j=1} y_{i(j)} \geq z \quad \text{and} \quad \Sigma^{l-1}_{j=1} y_{i(j)} \leq z-1$$

since z, **y** is a solution to *P.1*(φ)

This min path transforms to a min path composed of $\{i_1,...,i_l\}$ in φ_1 after contraction $K(G_{i(1)},...,G_{i(l)}|\varphi_o)$. We only need to show that this min path, after contraction, is not redundant in the presence of some other path in φ_1. Suppose to the contrary that there exists such a min path in φ_1 which consists of a subset of the components in $\{i_1,...,i_l\}$. Since it is the result of the contraction performed above, it must satisfy,

$$\Sigma^l_{j=1} y_{i(j)} - y_{i(k)} \geq z \quad \text{for some } k \in \{1,...,l\}$$

but this is not possible since it was assumed that

$$\Sigma^l_{j=1} y_{i(j)} - y_{i(k)} \leq z-1 \text{ for any } k \in \{1,...,l\}.$$

Therefore, the path $\{i_1, \ldots, i_l\}$ of φ_1 is indeed a min path of φ_1. Hence any min path of φ is a min path of φ_1. //

Theorem 6.3

If *C.1*(φ) has a solution w, **x**, then φ can be reached from a symmetric structure (namely the Σx_i-w+1/Σx_i structure) by contraction, K, and then omission of cuts, OC.

Proof: Parallel to that of Theorem 6.2. //

Remark

If the integer constraint is relaxed in *P.1*(φ) and a solution (z, **y**) obtained, then if it is not integer, we know that an integer solution to *P.1*(φ) also exists. To show this use the theory developed for linear programming. Imagine the convex polyhedron formed by the inequalities of *P.1*(φ). Basic solutions correspond to corner points and all other solutions, which correspond to the space within the polyhedron, are convex combinations of the basic solutions. Basic solutions are found as the result of linear operations on the inequalities of *P.1*(φ) whose coefficients are rational (in fact integer and non negative). Therefore, basic solutions are rational. Now if *P.1*(φ) has a solution it must have also some basic solution(s). Therefore, it has a rational solution and hence there is an integer ρ so that ρz, ρ**y** are integer and satisfy *P.1*(φ).

Proposition 6.4

If φ is derived from a k/N structure by (K, OP) then it is

derived from a $k/N+N_1$ structure ($N_1 \geq 0$) by (K, OP')

Proof:

$k/N+N_1$ contains all min paths of k/N (call the set of min paths of k/N, $\mathcal{P}_1$). It also has min paths that contain components from both N and N_1 (call this set of paths $\mathcal{P}_2$); and min paths that contain components from N_1 only (call this set of min paths $\mathcal{P}_3$).

Apply now contraction K on $k/N+N_1$. The min paths in $\mathcal{P}_1$ will be, after contraction, the same as when K is applied to k/N. This is true because no min path in $\mathcal{P}_1$ will become redundant in the presence of some min path contracted in $\mathcal{P}_2$ or $\mathcal{P}_3$ since $\mathcal{P}_2$ and $\mathcal{P}_3$ after contraction will contain components of N_1 ($\mathcal{P}_1$ does not) and because K is applied to components in N only.

Therefore, performing K on $k/N+N_1$ will result in a structure that contains all min paths of φ. By applying now some appropriate OP' we can reach φ. //

Proposition 6.5

If φ is derived from a k/N structure by (K, OC) then it is derived from a $k+N_1/N+N_1$ structure ($N_1 \geq 0$) by (K, OC')

Proof:

Use duality and Proposition 6.4:

If φ is derived from k/N by (K, OC) then φ^D is derived from $N-K+1/N$ by (K, OP). Using Proposition 6.4, φ^D is derived from $N-K+1/N+N_1$ by (K, OP') and thus by duality φ is derived from $k+N_1/N+N_1$ by (K, OC'). //

Proposition 6.6

If φ is derived from k/N by (K, OP) or (K, OC) then it is derived from ρK/ρN by (K', OP) or (K', OC) where ρ is integer positive.

Proof:

If z, $\mathbf{y}$ is a solution to *P.1*(φ) then so is any positive integer multiple ρz, $\rho\mathbf{y}$. //

Before we proceed with the next Theorems a figure will be helpful:

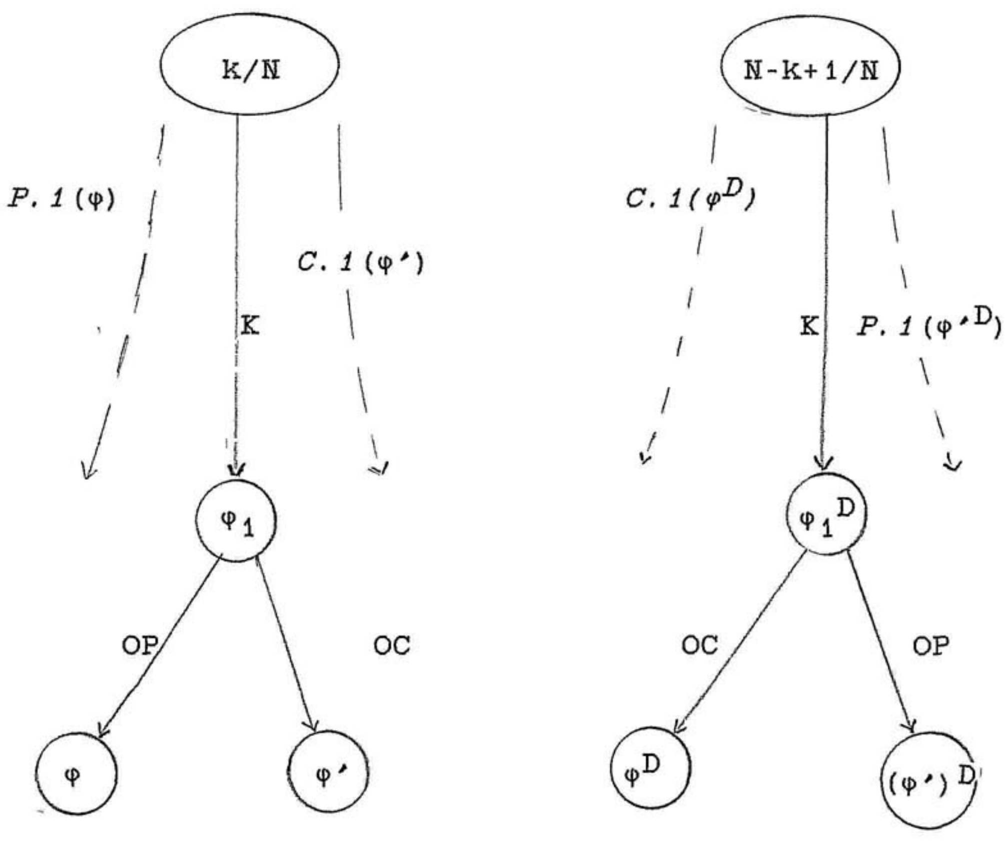

Figure 6.1a

Figure 6.1b

Figure 6.1a says that starting from a symmetric structure k/N, we perform some contraction K and reach a new structure φ_1. If OP is performed on φ_1 the structure φ is obtained and Problem *P.1*(φ) is the associated problem that searches for the symmetric "mother structure(s)" of φ one of which is k/N. If OC is performed on φ_1 another structure φ' is obtained. The associated Problem *C.1*(φ') searches for the symmetric "mother structure(s)" of φ' one of which is k/N

Figure 6.1b deals with the duals of the above where OC of Fig. 6.1b corresponds to the min paths of OP in Fig. 6.1a and OP of Fig. 6.1b corresponds to the min cuts of OC in Fig. 6.1a.

Theorem 6.7

P.1(φ) has a solution z, **y** iff *C.1*(φ^D) has a solution z, **y**.

Proof:

Refer to Figures 6.1a, 6.1b to visualize the proof.

z, **y** is a solution to *P.1*(φ), implies that φ is derived from a $z/\Sigma y_i$ structure by contraction, K, to reach φ_1 and then omission of paths, OP, to reach φ (by Theorem 6.2).

Take now $\Sigma y_i - z + 1/\Sigma y_i$ (it is the dual of $z/\Sigma y_i$) we apply the same contraction, K, as before. The resulting structure is φ_1^D, since each min path of $z/\Sigma y_i$ is a min cut of $\Sigma y_i - z + 1/\Sigma y_i$.

Applying now OP on φ_1 leads to φ and equivalently applying OC to φ_1^D (on the min paths of φ_1 which are now min cuts of φ_1^D) leads to φ^D.

Therefore, $C.1(\varphi^D)$ has a solution: $z, \mathbf{y}$.
The inverse follows trivially by symmetry arguments. //

Note:

When we say that $C.1(\varphi^D)$ has a solution $z, \mathbf{y}$, it means that the min cuts of the symmetric "mother structure" of φ^D can be found by taking all combinations of z components out of a total of Σy_i components. Therefore, the symmetric "mother structure" of φ^D is a $\Sigma y_i - z+1/\Sigma y_i$ structure.

Theorem 6.8

$C.1(\varphi)$ and $P.1(\varphi)$ have a common (dually related) solution $(\Sigma y_i - z+1, \mathbf{y})$ and $(z, \mathbf{y})$ respectively iff φ is derived from a symmetric structure by contraction, K, only.

Proof:

(i) Let φ be the result of contraction K only from k/N. then $P.1(\varphi)$ has a solution $z, \mathbf{y}$ $(z=k, \ \Sigma y_i = N)$. Now take the dual N-k+1/N symmetric structure and apply the same contraction K; we will arrive at φ^D. Therefore, $P.1(\varphi^D)$ has a solution; namely, $\Sigma y_i - z+1, \mathbf{y}$ $(\Sigma y_i - z+1 = N-k+1, \ \Sigma y_i = N)$.
It follows that $C.1(\varphi)$ has a solution, $\Sigma y_i - z+1, \mathbf{y}$ since $C.1(\varphi)$ is the same as $P.1(\varphi^D)$.

(ii) Let $P.1(\varphi)$ have a solution $z, \mathbf{y}$ then I can start with the structure $z/\Sigma y_i$, perform K to reach a new structure φ_1 and then perform OP to reach φ.
Also $C.1(\varphi)$ has a solution $\Sigma y_i - z+1, \mathbf{y}$. Then again I can start from the symmetric structure $z/\Sigma y_i$, apply K to reach φ_1 and then apply OC to reach φ.

Therefore, I can reach φ from φ_1 by either some OP or some OC. But this is impossible unless both OP and OC are vacuous operations.

Therefore, whenever $P.1(\varphi)$ and $C.1(\varphi)$ have a dually related solution $z, \mathbf{y}$ and $\Sigma y_i - z + 1, \mathbf{y}$ respectively, φ is derived by contraction alone from a $z/\Sigma y_i$ symmetric structure. //

6.2 On the Derivation of any Coherent Structure from some Symmetric Structure by OP (or OC) and then Contraction (K)

Up to this point we were concerned with demonstrating how the solution to $P.1(\varphi)$ or/and $C.1(\varphi)$ are related to φ being derived from a k/N structure by contraction, K, and then OP (or OC); or if $P.1(\varphi)$ and $C.1(\varphi)$ have a dually related solution, by contraction alone. Everything depends on the existence of solutions to $P.1(\varphi)$ or/and $C.1(\varphi)$ and hence there is no guarantee that we can reach any given structure φ from some symmetric "mother structure" by the sequence of operations K, OP (or OC)

In this section we are going to show that, when we are allowed to reverse the sequence of operations, i.e. perform first OP (or OC) and then K on a k/N structure, and when k/N, OP, K are appropriately chosen, any given coherent structure φ can be reached.

Consider the following problems,

$P.0(\varphi)$

$\mathbf{P}_\varphi \mathbf{y} \geq z\mathbf{e}$

$z \geq$ the number of components in the longest path of φ

$z, \mathbf{y}$ integer positive

$C.O(\varphi)$

$C_{\varphi}\mathbf{x} \geq we$

$w \geq$ the number of components in the longest cut of φ

$w, \mathbf{x}$ integer positive

Observe that both problems always have a solution.

Theorem 6.9

Let φ a coherent structure and let $P.O(\varphi)$ have solution $z, \mathbf{y}$. Then φ can be reached from a z out of Σy_i symmetric structure by the sequence of operations OP and K.

Proof:

Consider the z out of Σy_i structure and call it φ_o.

Pick non overlapping groups of components of φ_o:

G_1 contains y_1 components

G_2 contains y_2 components

. .

. .

G_n contains y_n components

Perform the contraction $K(G_1, \ldots, G_n | \varphi_o)$ but do not omit redundant paths for the moment. We only need to show that before omitting redundant paths any min path of φ is included.

Choose a min path of φ, say $P_i = \{i_1, \ldots, i_l\}$. This path can be derived by contraction from any min path of $z/\Sigma y_i$ that contains elements from each of the groups $G_{i(1)}, \ldots, G_{i(l)}$ and only from those. Indeed, there is such a min path in $z/\Sigma y_i$ because z is not less than the number of components in

any min path of φ, thus $z \geq 1$. It follows that because $z/\Sigma y_i$ contains all possible combinations of paths, it will contain paths with components from each of the groups $G_{i(1)}, \ldots, G_{i(l)}$. Further, since $\Sigma^{l}{}_{j=1} y_{i(j)} \geq z$ it follows that there is a path from those containing components from each of $G_{i(1)}, \ldots G_{i(l)}$, that contains components only from those groups. In other words, by picking components from each of the groups $G_{i(1)}, \ldots, G_{i(l)}$, I can form a set of z elements which is a min path of $z/\Sigma y_i$ since symmetric structures include all possible combinations.

We formed, therefore, for each min path of φ a corresponding min path of $z/\Sigma y_i$ that results to it when $K(G_1, \ldots$ $\ldots, G_n | \varphi_o)$ is applied and before omission of redundant paths. Therefore, by first omitting all other min paths of $z/\Sigma y_i$ and then performing $K(G_1, \ldots, G_n | \varphi_o)$ we will obtain φ. //

Remarks

1. The "dual" theorem holds also for $C.O(\varphi)$ and (OC, K).
2. As in Proposition 6.7 we may show that $P.O(\varphi)$ has a solution $z, \mathbf{y}$ iff $C.O(\varphi^D)$ has a solution $z, \mathbf{y}$
3. Out of all possible "mother structures" $z/\Sigma y_i$ (or $w/\Sigma x_i$) we might be interested in those that minimize Σy_i or minimize Σx_i. Then $P.O(\varphi)$, $C.O(\varphi)$ are slightly transformed to become integer linear programming problems,

$P.O'(\varphi)$

Min Σy_i

$\mathbf{P}_\varphi \mathbf{y} \geq z\mathbf{e}$

$\mathbf{y}$ integer positive

$C.O'(\varphi)$

Min Σx_i

$\mathbf{C}_\varphi \mathbf{x} \geq w\mathbf{e}$

$\mathbf{x}$ integer positive

where

z: is the number of components of the largest min path of φ

$=\max_i(r_i)$ where $\mathbf{r}=\mathbf{P}_\varphi \mathbf{e}$

w: the number of components of the largest min cut of φ

$=\max_i(s_i)$ where $\mathbf{s}=\mathbf{C}_\varphi \mathbf{e}$

Bounds on their optimal solution can be obtained as usual by relaxing the integer constraint and solving the corresponding linear program. To give it the standard form of a linear program we may define the variable $\mathbf{y}'=\mathbf{y}-\mathbf{e}$ and substitute.

6.3 On the Derivation of Self Dual Structures

Self dual structures have the property that any min cut is also a min path and the opposite. This property enables us to make the following statements,

Proposition 6.10

Let $\varphi \in \mathcal{M}$

(a) If φ is derived from a structure φ_o that belongs to $\mathcal{S}_2$ by (OP, K) then it is derivable from φ_o by (K, OP').

(b) If φ is derived from a φ_o structure that belongs to $\overline{\mathcal{S}}_2$ by (OC, K) then it is also derivable from φ_o by (K, OC')

Proof:

(a) φ is derived from φ_o by (OP, K). Let us apply K directly to φ_o, leading to φ_2. Schematically,

$$\begin{tikzcd}
 & \varphi_1 \arrow[r, "K"] & \varphi \in \mathcal{M} \\
\varphi_o \arrow[ur, "OP"] \arrow[dr, "K"'] & & \\
 & \varphi_2 &
\end{tikzcd}$$

Figure 6.2

Examine now φ_2. We will show that φ_2 contains all the min paths of φ. Hence, applying some OP' to φ_2 we are led to φ as required.

Case 1: OP omits only paths which are going to become redundant

anyway when K is applied directly to φ_o. Then, of course, $\varphi_2 = \varphi$

Case 2: OP omits some min paths which if not omitted they would not become redundant if K is applied directly to φ_o. This means that these min paths will appear in φ_2 but not in φ.

There are two subcases:

Case 2.1: There are min paths in φ that do not appear in φ_2. Because of assumption of Case 2, any such min path in φ does not appear in φ_2 because it is redundant in the presence of some min path in φ_2. This means that for each such min path P_i of φ, there is a min path in φ_2 say P_{2j}, so that $P_i \supset P_{2j}$. But by assumption φ_o belongs to $\boldsymbol{S}_2$ and thus $\varphi_2 \in \boldsymbol{S}_2$. Therefore, P_{2j} is a cut (perhaps not a min cut) for the min paths of φ_2. But each min path of φ_2 is also a min path of φ except perhaps for some min paths of φ for which $P_i \supset P_{2j}$, as we discussed above. Then P_{2j} is also a cut for those min paths since each of these contains the components of a min path of φ_2. Therefore, P_{2j} is a cut of φ. But $\varphi \in \boldsymbol{M}$ and P_i which is a min path of φ, is also a min cut of φ. But we found a cut of φ, namely P_{2j}, so that $P_i \supset P_{2j}$. Contradiction. Therefore, Case 2.1 is not possible.

Case 2.2: (The negation of Case 2.1): All min paths of φ are also min paths of φ_2. Then, of course there is an operation OP' that leads us from φ_2 to φ.

(b) the proof proceeds by arguing in parallel to (a) above replacing cuts for paths and $\overline{S}_2$ for S_2. //

Proposition 6.11

Let $\varphi \in \mathcal{M}$. If φ is obtained from a k/n structure by contraction alone, then φ is obtained from the n-k+1/n structure by the same contraction alone.

Proof:

By applying K to k/n we obtain φ by assumption. We know that by applying K to n-k+1/n we obtain φ^D since the min paths of k/n are the min cuts of n-k+1/n. But since $\varphi \in \mathcal{M}$, $\varphi = \varphi^D$. //

Proposition 6.12

Let $\varphi \in \mathcal{M}$. If φ is obtained from k/n by contraction alone then it is obtained by the same contraction alone from any k'/n structure with $\min(k, n-k+1) \le k' \le \max(k, n-k+1)$.

Proof:

By Proposition 6.11 φ is obtained by K alone from either k/n or n-k+1/n. Let the contraction K be given by $K(\mathbf{y} | \varphi_{k/n})$ where $\Sigma y_i = n$. Then both $(k, \mathbf{y})$ and $(\Sigma y_i - k+1, \mathbf{y})$ are solutions of *P. 1* (φ). Therefore,

$\mathbf{P}_\varphi \mathbf{y} \ge k\mathbf{e}$

$\mathbf{P}_{\varphi}' \mathbf{y} \le (k-1)\mathbf{e}$

and

$\mathbf{P}_\varphi \mathbf{y} \ge (\Sigma y_i - k+1)\mathbf{e}$

$\mathbf{P}_{\varphi}' \mathbf{y} \le (\Sigma y_i - k)\mathbf{e}$

But then $\mathbf{P}_\varphi \mathbf{y} \ge k'\mathbf{e}$ for any $k' \ge \min(k, \Sigma y_i - k+1)$

and $\mathbf{P}_{\varphi}' \mathbf{y} \le (k'-1)\mathbf{e}$ for any $k' \le \max(k, \Sigma y_i - k+1)$ //

Remark

A bound on the range of k′, as determined in Proposition 6.12, can be established. Using results of Chapter 7 (Theorem 7.3 and 7.8) we know that for a k/n structure to belong to $\mathcal{S}_2$ but not to $\mathcal{S}_3$ k must satisfy,

$$(n+1)/2 \leq k < (2n+1)/3$$

while for k/n to belong to $\overline{\mathcal{S}}_2$ but not $\overline{\mathcal{S}}_3$ k must satisfy

$$(n+1)/2 \leq n-k+1 < (2n+1)/3$$

Now if a structure belongs to $\mathcal{S}_3$ contraction will keep it in $\mathcal{S}_3$ and the same for $\overline{\mathcal{S}}_3$. Therefore, if we are to reach an $\varphi \in \mathcal{M}=\mathcal{S}_2 \cap \overline{\mathcal{S}}_2$ by contraction only, we have to start from some k/n structure in $\mathcal{S}_2$ (but not in $\mathcal{S}_3$) or $\overline{\mathcal{S}}_2$ (but not in $\overline{\mathcal{S}}_3$). Therefore, k must be limited in the range $(n+2)/3<k<(2n+1)/3$ which is a bound for the range of k′ as found in Proposition 6.12.

Example 6.13

We want to form a committee to decide on the public works that are to be undertaken in the coming years. In this committee we wish the following people to be represented,

(a) The local authorities

(b) The association of engineers

(c) The association of urban planers

(d) The central government

(e) The financial institutions

Further a decision passes if any one of the following min paths passes it:

Path 1: a, b

Path 2: a, c

Path 3: a, d, e

Path 4: b, c, d

Path 5: b, c, e

where a, b, c d, e refer to the groups of people described above.

Is the structure φ as described by the paths above going to give consistent answers?

Check to see that any two paths have a common component and therefore $\varphi \in \mathcal{S}_2$. Further, investigation reveals that any min path is also a min cut (and the opposite) and therefore the structure belongs to $\boldsymbol{M}$ (it is self dual).

The structure φ, therefore, is never blocked or contradictory but it cannot support the AND or OR conjunction or disjunction of statements in the outcome sets because this requires the limitation of the set of alternatives A to $A \leq 2$ and $|\mathcal{A}_i| = 1$ as was discussed in Chapter 4.

Is it possible to represent φ by a k/n structure?

To find a k/n "mother structure" of φ, if there is one, we must solve problem $P.1(\varphi)$ (or $C.1(\varphi)$ since $\varphi \in \boldsymbol{M}$)

In our case $\mathbf{P}_\varphi$ is given by,

$$\mathbf{P}_\varphi = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

There is a solution $z, \mathbf{y}$ to

$\mathbf{P}_\varphi\ \mathbf{y} = z\ \mathbf{e}$ where $\mathbf{y} = (y_a, y_b, y_c, y_d, y_e)$

namely, $z=5$, $\mathbf{y} = (3, 2, 2, 1, 1)$ and therefore the $z/\Sigma y_i$ structure is the 5/9 structure where

(a) The local authorities have three votes or representatives that vote always as a group ($y_a=3$)

(b) The association of engineers has two representatives ($y_b=2$)

(c) The association of urban planners has 2 representatives ($y_c=2$)

(d) The central government has 1 ($y_d=1$)

(e) The financial institutions have 1 ($y_e=1$).

Note that when $\mathbf{P}_\varphi\mathbf{y}=z\mathbf{e}$ is satisfied then so are the constraints of $P.1(\varphi)$

Example 6.14

Consider the structure φ containing 4 components, whose min paths are , $\{1,2\}$, $\{1,3,4\}$, $\{2,3\}$, $\{3,4\}$. First check that each min path is also a min cut and that φ is self dual. It easy to check that φ can be derived from

(a) a 3/5 symmetric structure by contraction only. ($G_1=\{1\}$, $G_2=\{2,3,4\}$, $G_3=\{4\}$, $G_4=\{5\}$).

(b) a 4/7 symmetric structure by contraction only ($G_1=\{1\}$, $G_2=\{2,3,4\}$, $G_3=\{5\}$, $G_4=\{6,7\}$).

(c) a 5/9 symmetric structure by contraction only ($G_1=\{1,2\}$, $G_2=\{3,4,5\}$, $G_3=\{6,7\}$, $G_4=\{8,9\}$).

(d) a 6/11 symmetric structure by contraction only ($G_1=\{1,2,3\}$, $G_2=\{4,5,6,7\}$, $G_3=\{8,9\}$, $G_4=\{10,11\}$).

(e) a 6/12 or a 7/12 symmetric structure by contraction only ($G_1=\{1,2,3\}$, $G_2=\{4,5,6,7,8\}$, $G_3=\{9,10\}$, $G_4=\{11,12\}$). This complies with Proposition 6.12.

(f) a 7/15 or a 8/15 or a 9/15 symmetric structure by contraction only ($G_1=\{1,2,3\}$, $G_2=\{4,5,6,7,8,9\}$, $G_3=\{10,11,12\}$, $G_4=\{13,14,15\}$).

(g) Because of Proposition 6.6 and since 7/15, 8/15, 9/15 are mother structures of φ so are the structures 14/30, 16/30, 18/30. Further since the operation of $\dot{O}P$ is vacuous for 7/15, 8/15, 9/15 mother structures, so is for the mother structures 14/30, 16/30, 18/30 from which φ is derived by contraction K only. But by Proposition 6.12 the range of mother structures can be extended to become 14/30, 15/30, 16/30, 17/30, 18/30 from which φ is derived by the same contraction K alone. Also observe that k takes values within the range

$$32/3=(n+2)/3<k<(2n+1)/3=61/3$$

as we discussed in the remark following Proposition 6.12.

A question naturally arises at this point: Given a self dual structure φ, is there always some k/n mother structure from which φ can be derived by contraction alone? A counter example (Example 6.15) shows that this is not true.

Example 6.15

A decision structure φ is formed so that it is a 2/3 structure whose 3 components (a, b, c) are themselves odd majority structures:

component a is a 2/3 structure (contains components 1, 2, 3)

component b is a 2/3 structure (contains components 4, 5, 6)

component c is a 3/5 structure (contains components 7, 8, 9, 10, 11)

Solving *P.1*(φ) we see that the 11 component structure above, is the result of a contraction from a 12/28 structure where components 1, 2, 3, 4, 5, 6 represent three original components of the "mother structure" each ; and components 7, 8, 9, 10, 11 represent two original components each in the 12/28 "mother structure" i.e. $y_1=y_2=\ldots=y_6=3$, $y_7=\ldots=y_{11}=2$. Further, after the contraction according to $\mathbf{y}$ is performed on 12/28, we need OP to reach φ otherwise min paths containing components 1, 2, 3, for example, would appear after contraction; while they do not appear in φ.

Example 6.16

We want to find the mother structure of a structure φ with 9 components , whose min paths are described by $\mathbf{P}_\varphi$, so that the mother structure has the minimum possible number of components. *P.1*(φ) was solved as a linear program relaxing the integrality requirement while adding the objective function Min Σy_i. The mother structure that does the job is the 6/15 symmetric structure. The formulation as well as the optimal solution which is fortunately integer follows,

```
:
MIN      Y1 + Y2 + Y3 + Y4 + Y5 + Y6 + Y7 + Y8 + Y9
SUBJECT TO
:       2)    Y1 + Y2 + Y5 + Y6 + Y7 - Z >=    0
:       3)    Y2 + Y3 + Y5 - Z >=    0
:       4)    Y3 + Y4 + Y5 + Y9 - Z >=    0
:       5)    Y4 + Y8 + Y9 - Z >=    0
:       6)    Y1 + Y5 + Y8 - Z >=    0
        7)    Y6 + Y7 + Y8 - Z >=    0
        8)    Y1 >=    1
        9)    Y2 >=    1
       10)    Y3 >=    1
       11)    Y4 >=    1
       12)    Y5 >=    1
       13)    Y6 >=    1
       14)    Y7 >=    1
       15)    Y8 >=    1
       16)    Y9 >=    1
       17)    Z >=    1
       18)    Y1 + Y2 + Y5 + Y6 - Z <= - 1
       19)    Y1 + Y2 + Y5 + Y7 - Z <= - 1
       20)    Y1 + Y2 + Y6 + Y7 - Z <= - 1
       21)    Y1 + Y5 + Y6 + Y7 - Z <= - 1
       22)    Y2 + Y5 + Y6 + Y7 - Z <= - 1
       23)    Y3 + Y5 - Z <= - 1
--More--
       24)    Y2 + Y5 - Z <= - 1
       25)    Y2 + Y3 - Z <= - 1
       26)    Y4 + Y5 + Y9 - Z <= - 1
       27)    Y3 + Y5 + Y9 - Z <= - 1
       28)    Y3 + Y4 + Y9 - Z <= - 1
       29)    Y3 + Y4 + Y5 - Z <= - 1
       30)    Y8 + Y9 - Z <= - 1
       31)    Y4 + Y9 - Z <= - 1
       32)    Y4 + Y8 - Z <= - 1
       33)    Y5 + Y8 - Z <= - 1
       34)    Y1 + Y8 - Z <= - 1
       35)    Y1 + Y5 - Z <= - 1
       36)    Y7 + Y8 - Z <= - 1
       37)    Y6 + Y8 - Z <= - 1
       38)    Y6 + Y7 - Z <= - 1
END
          OBJECTIVE FUNCTION VALUE

 1)          15.0000000

 VARIABLE          VALUE            REDUCED COST
       Y1         1.000000              .000000
       Y2         2.000000              .000000
       Y3         3.000000              .000000
       Y4         1.000000              .000000
       Y5         1.000000              .000000
       Y6         1.000000              .000000
       Y7         1.000000              .000000
       Y8         4.000000              .000000
       Y9         1.000000              .000000
        Z         6.000000              .000000
```

```
MIN      Y1 + Y2 + Y3 + Y4 + Y5 + Y6 + Y7 + Y8 + Y9
SUBJECT TO
       2)    Y1 + Y2 + Y5 + Y6 + Y7 - Z >=    0
       3)    Y2 + Y3 + Y5 - Z >=    0
       4)    Y3 + Y4 + Y5 + Y9 - Z >=    0
       5)    Y4 + Y8 + Y9 - Z >=    0
       6)    Y1 + Y5 + Y8 - Z >=    0
       7)    Y6 + Y7 + Y8 - Z >=    0
       8)    Y1 >=    1
       9)    Y2 >=    1
      10)    Y3 >=    1
      11)    Y4 >=    1
      12)    Y5 >=    1
      13)    Y6 >=    1
      14)    Y7 >=    1
      15)    Y8 >=    1
      16)    Y9 >=    1
      17)    Z >=    1
      18)    Y1 + Y2 + Y5 + Y6 - Z <= - 1
      19)    Y1 + Y2 + Y5 + Y7 - Z <= - 1
      20)    Y1 + Y2 + Y6 + Y7 - Z <= - 1
      21)    Y1 + Y5 + Y6 + Y7 - Z <= - 1
      22)    Y2 + Y5 + Y6 + Y7 - Z <= - 1
      23)    Y3 + Y5 - Z <= - 1
--More--
      24)    Y2 + Y5 - Z <= - 1
      25)    Y2 + Y3 - Z <= - 1
      26)    Y4 + Y5 + Y9 - Z <= - 1
      27)    Y3 + Y5 + Y9 - Z <= - 1
      28)    Y3 + Y4 + Y9 - Z <= - 1
      29)    Y3 + Y4 + Y5 - Z <= - 1
      30)    Y8 + Y9 - Z <= - 1
      31)    Y4 + Y9 - Z <= - 1
      32)    Y4 + Y8 - Z <= - 1
      33)    Y5 + Y8 - Z <= - 1
      34)    Y1 + Y8 - Z <= - 1
      35)    Y1 + Y5 - Z <= - 1
      36)    Y7 + Y8 - Z <= - 1
      37)    Y6 + Y8 - Z <= - 1
      38)    Y6 + Y7 - Z <= - 1
      39)    Y5 + Y7 + Y8 - Z >=    0
      40)    Y7 + Y8 - Z <= - 1
      41)    Y5 + Y7 - Z <= - 1
      42)    Y5 + Y8 - Z <= - 1
      43)    Y2 + Y3 + Y8 + Y9 - Z >=    0
      44)    Y3 + Y8 + Y9 - Z <= - 1
      45)    Y2 + Y8 + Y9 - Z <= - 1
      46)    Y2 + Y3 + Y9 - Z <= - 1
      47)    Y2 + Y3 + Y8 - Z <= - 1

   NO FEASIBLE SOLUTION AT STEP       8
```

Example 6.18

The simplex algorithm was applied to Min Σy_i subject to the constraints of *P.1*(φ) with φ having 20 components and relaxing the integrality constraint. The optimal solution was non integer. By multiplying the solution by 2 we obtain an integer solution to *P.1*(φ), and the mother structure is 24/64.

The computer formulation and solution appears below.

```
MIN      Y1 + Y2 + Y3 + Y4 + Y5 + Y6 + Y7 + Y8 + Y9 + Y10 + Y11 + Y12
    + Y13 + Y14 + Y16 + Y17 + Y18 + Y19 + Y20
SUBJECT TO
        2)  Y1 + Y2 + Y3 + Y5 + Y7 + Y9 + Y10 + Y13 + Y16 + Y15 - Z >= 0
        3)  Y2 + Y3 + Y5 + Y7 + Y9 + Y10 + Y13 + Y16 + Y15 - Z <= - 1
        4)  Y1 + Y3 + Y5 + Y7 + Y9 + Y10 + Y13 + Y16 + Y15 - Z <= - 1
        5)  Y1 + Y2 + Y5 + Y7 + Y9 + Y10 + Y13 + Y16 + Y15 - Z <= - 1
        6)  Y1 + Y2 + Y3 + Y7 + Y9 + Y10 + Y13 + Y16 + Y15 - Z <= - 1
        7)  Y1 + Y2 + Y3 + Y5 + Y9 + Y10 + Y13 + Y16 + Y15 - Z <= - 1
        8)  Y1 + Y2 + Y3 + Y5 + Y7 + Y10 + Y13 + Y16 + Y15 - Z <= - 1
        9)  Y1 + Y2 + Y3 + Y5 + Y7 + Y9 + Y13 + Y16 + Y15 - Z <= - 1
       10)  Y1 + Y2 + Y3 + Y5 + Y7 + Y9 + Y10 + Y16 + Y15 - Z <= - 1
       11)  Y1 + Y2 + Y3 + Y5 + Y7 + Y9 + Y10 + Y13 + Y16 - Z <= - 1
       12)  Y1 + Y2 + Y3 + Y5 + Y7 + Y9 + Y10 + Y13 + Y15 - Z <= - 1
       13)  Y2 + Y5 + Y8 + Y9 + Y10 + Y11 + Y19 + Y20 - Z >= 0
       14)  Y5 + Y8 + Y9 + Y10 + Y11 + Y19 + Y20 - Z <= - 1
       15)  Y2 + Y8 + Y9 + Y10 + Y11 + Y19 + Y20 - Z <= - 1
       16)  Y2 + Y5 + Y9 + Y10 + Y11 + Y19 + Y20 - Z <= - 1
       17)  Y2 + Y5 + Y8 + Y10 + Y11 + Y19 + Y20 - Z <= - 1
       18)  Y2 + Y5 + Y8 + Y9 + Y11 + Y19 + Y20 - Z <= - 1
       19)  Y2 + Y5 + Y8 + Y9 + Y10 + Y19 + Y20 - Z <= - 1
       20)  Y2 + Y5 + Y8 + Y9 + Y10 + Y11 + Y20 - Z <= - 1
       21)  Y2 + Y5 + Y8 + Y9 + Y10 + Y11 + Y19 - Z <= - 1
       22)  Y1 + Y2 + Y11 + Y12 + Y14 + Y17 + Y15 - Z >= 0
--More--
       23)  Y2 + Y11 + Y12 + Y14 + Y17 + Y15 - Z <= - 1
       24)  Y1 + Y11 + Y12 + Y14 + Y17 + Y15 - Z <= - 1
       25)  Y1 + Y2 + Y12 + Y14 + Y17 + Y15 - Z <= - 1
       26)  Y1 + Y2 + Y11 + Y14 + Y17 + Y15 - Z <= - 1
       27)  Y1 + Y2 + Y11 + Y12 + Y17 + Y15 - Z <= - 1
       28)  Y1 + Y2 + Y11 + Y12 + Y14 + Y17 - Z <= - 1
       29)  Y1 + Y2 + Y11 + Y12 + Y14 + Y15 - Z <= - 1
       30)  Y1 + Y5 + Y6 + Y18 + Y19 + Y20 - Z >= 0
       31)  Y5 + Y6 + Y18 + Y19 + Y20 - Z <= - 1
       32)  Y1 + Y6 + Y18 + Y19 + Y20 - Z <= - 1
       33)  Y1 + Y5 + Y18 + Y19 + Y20 - Z <= - 1
       34)  Y1 + Y5 + Y6 + Y19 + Y20 - Z <= - 1
       35)  Y1 + Y5 + Y6 + Y18 + Y20 - Z <= - 1
       36)  Y1 + Y5 + Y6 + Y18 + Y19 - Z <= - 1
```

```
      37)    Y3 + Y4 + Y5 + Y9 + Y10 + Y11 + Y14 + Y18 + Y15 - Z >= 0
      38)    Y4 + Y5 + Y9 + Y10 + Y11 + Y14 + Y18 + Y15 - Z <= - 1
      39)    Y3 + Y5 + Y9 + Y10 + Y11 + Y14 + Y18 + Y15 - Z <= - 1
      40)    Y3 + Y4 + Y9 + Y10 + Y11 + Y14 + Y18 + Y15 - Z <= - 1
      41)    Y3 + Y4 + Y5 + Y10 + Y11 + Y14 + Y18 + Y15 - Z <= - 1
      42)    Y3 + Y4 + Y5 + Y9 + Y11 + Y14 + Y18 + Y15 - Z <= - 1
      43)    Y3 + Y4 + Y5 + Y9 + Y10 + Y14 + Y18 + Y15 - Z <= - 1
      44)    Y3 + Y4 + Y5 + Y9 + Y10 + Y11 + Y18 + Y15 - Z <= - 1
      45)    Y3 + Y4 + Y5 + Y9 + Y10 + Y11 + Y14 + Y18 - Z <= - 1
      46)    Y3 + Y4 + Y5 + Y9 + Y10 + Y11 + Y14 + Y15 - Z <= - 1
--More--
      47)    Y7 + Y9 + Y12 + Y13 + Y16 - Z >= 0
      48)    Y9 + Y12 + Y13 + Y16 - Z <= - 1
      49)    Y7 + Y12 + Y13 + Y16 - Z <= - 1
      50)    Y7 + Y9 + Y13 + Y16 - Z <= - 1
      51)    Y7 + Y9 + Y12 + Y16 - Z <= - 1
      52)    Y7 + Y9 + Y12 + Y13 - Z <= - 1
      53)    Y1 >=   1
      54)    Y2 >=   1
      55)    Y3 >=   1
      56)    Y4 >=   1
      57)    Y5 >=   1
      58)    Y6 >=   1
      59)    Y7 >=   1
      60)    Y8 >=   1
      61)    Y9 >=   1
      62)    Y10 >=   1
      63)    Y11 >=   1
      64)    Y12 >=   1
      65)    Y13 >=   1
      66)    Y14 >=   1
      67)    Y15 >=   1
      68)    Y16 >=   1
      69)    Y17 >=   1
      70)    Y18 >=   1
--More--
      71)    Y19 >=   1
      72)    Y20 >=   1
      73)    Z >=   1
END
INTEGER-VARIABLES=       21


        OBJECTIVE FUNCTION VALUE

1)          32.0000000

VARIABLE          VALUE           REDUCED COST
      Y1         1.000000            .000000
      Y2         1.000000            .000000
      Y3         1.000000            .000000
      Y4         1.000000            .000000
```

Y5	1.000000	.000000
Y6	1.000000	.000000
Y7	2.500000	.000000
Y8	1.000000	.000000
Y9	1.500000	.000000
Y10	1.000000	.000000
Y11	1.000000	.000000
Y12	6.000000	.000000
Y13	1.000000	.000000
Y14	1.000000	.000000
Y16	1.000000	.000000
Y17	1.000000	.000000
Y18	3.500000	.000000
Y19	1.000000	.000000
Y20	4.500000	.000000
--More--		
Y15	1.000000	.000000
Z	12.000000	.000000

BOOK 3

ANSWERS AND LOGIC

Chapter 7

CONDITIONS FOR ANSWERS THAT OBEY LOGIC: SYMMETRIC STRUCTURES

7.1 Conditions for a Structure to Belong to $\mathcal{S}_p$ or $\overline{\mathcal{S}}_p$

For a coherent structure φ with m min paths let the following quantities be defined,

k_i: number of components in the ith min path of φ.

n_Q: number of different components in the p-plet of min paths defined by the set Q of p indices.

Q: a set of p indices from $\{1,...,m\}$.

l_i: number of components in the ith min cut of φ.

Theorem 7.1

Let a coherent structure φ.

If $\Sigma_{i\in Q}k_i \geq (p-1)n_Q+1$, for all Q then $\varphi\in\mathcal{S}_p$.

Proof:

Take any p min paths, $Q=\{1,2,...,p\}$. Let

r_{ij}: the number of components common between min paths $i,j\in Q$ and only those.

r_{ii}: the number of components that appear in min path i only.

$r_{i_1i_2i_3}$: the number of components common among min paths $i_1,i_2,i_3\in Q$ and only those $(i_1\neq i_2\neq i_3)$

Similarly we define $r_{i_1i_2i_3i_4},...,r_{i_1...i_p}=r_Q$

r_Q: the number of components common among all p min paths that belong to the set Q.

The following equations must be satisfied (one for each min path j in the p-plet Q)

$$r_{jj}+\sum_{i=1}^{p} r_{ji}+(1/2)\sum_{i_1,i_2\in Q} r_{ji_1i_2}+(1/3!)\sum_{i_1,i_2,i_3\in Q} r_{ji_1i_2i_3}+\ldots+r_Q=k_j \qquad \text{for all } j\in Q \tag{7.1}$$

where $i\neq j$ in the first summation, $j\neq i_1\neq i_2$ in the second, $j\neq i_1\neq i_2\neq i_3$ in the third etc.

Equation (7.1) says that the number of components in min path j is equal to k_j.

Also

$$\sum_{j\in Q} r_{jj}+(1/2!)\sum_{j,i\in Q} r_{ji}+(1/3!)\sum_{j,i_1,i_2\in Q} r_{ji_1i_2}+\ldots+r_Q=n_Q \tag{7.2}$$

where $j\neq i$ in the second summation, $j\neq i_1\neq i_2$ in the third etc.

Equation (7.2) says that the total number of different components in the p-plet defined by Q is equal to n_Q. Adding now the first p equations (7.1) and subtracting p-1 times equation (7.2) we obtain,

$$-(p-2)\sum_{j\in Q} r_{jj}-(p-3)(1/2!)\sum_{j,i\in Q} r_{ji}-(p-4)(1/3!)\sum_{j,i_1,i_2\in Q} r_{ji_1i_2}-\ldots$$

$$-\ldots+r_Q=\sum_{j\in Q} k_j-(p-1)n_Q \tag{7.3}$$

where $j\neq i$ in the second summation, $j\neq i_1\neq i_2$ in the third etc. and where we observe that $h/h!=1/(h-1)!$ for any positive integer h.

We claim that for the p min paths in Q to have at least one

common component it is sufficient that

$$\sum_{j\in Q} k_j-(p-1)n_Q\geq 1$$

Indeed this condition says that the RHS of (7.3) must be greater or equal to 1. But this, in turn implies that $r_Q\geq 1$, since all other terms in the LHS of (7.3) are non positive. But the condition $r_Q\geq 1$ is equivalent to requiring that all min paths in Q have at least one common component. //

Corollary 7.2

Let φ be a k out of n structure.

If $pk\geq(p-1)n+1$ then $\varphi\in\mathcal{S}_p$.

Proof:

By theorem 7.1 we know that for any p-plet of min paths (say the p-plet described by the set of indices Q),

$$\Sigma_{i\in Q}k_i\geq(p-1)n_Q+1 \quad \text{for all } Q \tag{7.4}$$

is sufficient for $\varphi\in\mathcal{S}_p$

Now by assumption

$$pk\geq(p-1)n+1 \tag{7.5}$$

also $pk=\Sigma k_i$ since φ is k out of n, while $n\geq n_Q$ and therefore (7.4) is proved from (7.5). //

The previous Corollary is now strengthened,

Theorem 7.3

Let φ be a k out of n structure. Then,

$$\varphi\in\mathcal{S}_p \iff kp\geq(p-1)n+1$$

Proof:

(a) Sufficiency:

$kp \geq (p-1)n+1 \Rightarrow \varphi \in \mathcal{S}_p$

because of Corollary 7.2

(b) Necessity: $\varphi \in \mathcal{S}_p \Rightarrow kp \geq n(p-1)+1$

The proof proceeds by examining cases where n takes successively the values below,

Case 1: $n=p\rho$ Thus, we have to show that

$$k \geq (p-1)\rho+1/p \Rightarrow k \geq (p-1)\rho+1$$

Case 2: $n=p\rho+1$ Thus, $k \geq (p-1)\rho+1$ is to be proved

Case 3: $n=p\rho+2$ Thus, $k \geq (p-1)\rho+2$ is to be proved

. .

. .

Case p: $n=p\rho+p-1$ Thus, $k \geq (p-1)\rho+p-1$ is to be proved

where ρ is positive integer. We now assume that the theorem is violated hoping to be led to contradiction.

Case 1:

Let $n=p\rho$. Suppose to the contrary, that $k \geq (p-1)\rho+1$ is not satisfied but instead $k=(p-1)\rho$. If k is even less than $(p-1)\rho$ so much the better towards our aim for contradiction. (See note below).

Consider groups of components $G_1, G_2, \ldots, G_p$ that are mutually exclusive and so that each one contains ρ components. Then $|G_1 \cup G_2 \cup \ldots \cup G_p| = p\rho = n$

Now construct the following min paths,

(1) $G_1 \cup G_2 \cup \ldots \cup G_{p-2} \cup G_{p-1}$ (G_p is missing)

(2) $G_2 \cup G_3 \cup \ldots \cup G_{p-1} \cup G_p$ (G_1 is missing)

. .

. .

(p) $G_p \cup G_1 \cup \ldots \cup G_{p-3} \cup G_{p-2}$ (G_{p-1} is missing)

These p paths have $(p-1)\rho = k$ components each and therefore, they are min paths of a k out of n structure (k and n as chosen above). But they have no common component among all p of them. Contradiction.

Note: If $k < (p-1)\rho$ repeat the proof but construct the p paths with less components each so that there length equals k. Still they are not going to have a common component.

Case 2:

Let $n = p\rho + 1$ and let to the contrary $k = (p-1)\rho$. Construct groups, $G_1, \ldots, G_p$ with ρ components each. Then

$|G_1 \cup \ldots \cup G_p| = p\rho = n$

Construct the same min paths as in Case 1. These p min paths have $(p-1)\rho$ components each and they are min paths of a k out of n structure with $k = (p-1)\rho$ and $n = p\rho$. But they have no common component among all p of them. Contradiction.

Case 3:

Let $n = p\rho + 2$ and let to the contrary $k = (p-1)\rho + 1$. Now construct groups:

G_1: $\rho + 1$ components

$G_2, G_3, \ldots, G_p$ each with ρ components

$\tilde{G}$ with one component.

Again $|G_1 \cup G_2 \cup \ldots \cup G_p \cup \tilde{G}| = p\rho + 2 = n$

Construct p paths as follows,

(1) $G_1 \cup G_2 \cup \ldots \cup G_{p-1}$ (G_p and $\tilde{G}$ are missing)

(2) $G_2 \cup G_3 \cup \ldots \cup G_p \cup \tilde{G}$ (G_1 is missing)

(3) $G_3 \cup \cup \ldots \cup G_p \cup G_1$ (G_2 and $\tilde{G}$ are missing)

. .

.   .

(p) $G_p \cup G_1 \cup \ldots \cup G_{p-3} \cup G_{p-2}$ (G_{p-1} and $\tilde{G}$ are missing)

Each of these min paths has $(p-1)\rho+1=k$ components and they are among the min paths of a k out of n structure with k and n as defined in this Case. But they have no common component among all p of them. Contradiction.

.

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Case l:

Let $n=p\rho+l$ and let to the contrary $k=(p-1)\rho+l-1$. Construct groups,

$G_1, G_2, \ldots, G_{l-1}$ with $\rho+1$ components each.

$G_l, G_{l+1}, \ldots, G_p$ with ρ components each.

$\tilde{G}$ with one component.

Construct min paths,

(1) $G_1 \cup G_2 \cup \ldots \cup G_{l-1} \cup G_l \cup \ldots \cup G_{p-1}$

(G_p and $\tilde{G}$ are missing)

(2) $G_2 \cup G_3 \cup \ldots \cup G_{l-1} \cup G_l \cup \ldots \cup G_p \cup \tilde{G}$

(G_1 is missing)

(3) $G_3 \cup G_4 \cup \ldots \cup G_{l-1} \cup G_l \cup \ldots \cup G_p \cup G_1 \cup \tilde{G}$

(G_2 is missing)

.

.

(l) $G_l \cup G_{l+1} \cup \ldots \cup G_p \cup G_1 \cup \ldots \cup G_{l-2} \cup \tilde{G}$

(G_{l-1} is missing)

(l+1) $G_{l+1} \cup G_{l+2} \cup \ldots \cup G_p \cup G_1 \cup \ldots \cup G_{l-1}$

(G_l and $\tilde{G}$ are missing)

.

.

(p) $G_p \cup G_1 \cup \ldots \cup G_{l-1} \cup G_l \cup \ldots \cup G_{p-2}$

(G_{p-1} and $\tilde{G}$ are missing)

Each path contains $(p-1)\rho+(l-1)=k$ components while $|G_1 \cup G_2 \cup \ldots \cup G_p \cup \tilde{G}|=p\rho+l=n$. Therefore, these p paths are min paths of a k out of n structure with k and n as chosen in this Case, but they have no common component among all p of them. Contradiction.

Therefore, $\varphi \in \mathcal{S}_p \Rightarrow kp \geq (p-1)n+1$ //

Remark

Similar ideas are presented by Ferejohn and Grether [1974] where they use the symbol α to mean k/n, while their (m-1)/m corresponds to our (p-1)/p approximately. Also in Peleg [1978] similar ideas are investigated and the condition $k \geq [n(p-1)/p]+1$ is found. Here, however, the approach is based on structural properties only and is not concerned with preference orderings of components.

Corollary 7.4

Let φ be a k out of n structure. Then,

$[\varphi \in \mathcal{S}_p$ and $\varphi \notin \mathcal{S}_{p+1}] \Leftrightarrow ((p-1)n+1)/p \leq k < (pn+1)/(p+1)$

Proof:

Use Theorem 7.3 //

Example 7.5

The 15 out of 20 structure belongs to S_3 but not to S_4.

The 14 out of 20 structure belongs to S_3.

The 16 out of 20 structure belongs to S_4.

The "dual" of Theorem 7.1 is

Theorem 7.6

Let a coherent structure φ.

If

$$\Sigma_{i\in Q} l_i \geq (p-1) n_Q + 1 \quad \text{for all } Q$$

then $\varphi \in \overline{S}_p$

(Recall that l_i is the length of the ith min cut, and Q is any set of p indices of min cuts)

Proof: As in Theorem 7.1 for min paths. //

Corollary 7.7

Let φ be a k out of n structure.

If,

$$pk \leq n-1+p$$

then $\varphi \in \overline{S}_p$

Proof:

By Theorem 7.6 we know that $\Sigma_{i\in Q} l_i \geq (p-1) n_Q + 1$. Since each cut has n-k+1 components, it follows that $p(n-k+1) \geq (p-1) n_Q + 1$ and therefore, $p(n-k+1) \geq (p-1)n+1$ is also a sufficient condition. It follows that $pk \leq n-1+p$ is a sufficient condition for $\varphi \in \overline{S}_p$ //

Theorem 7.8

Let φ be a k out of n structure. Then,

$$\varphi \in \overline{\mathcal{S}}_p \iff kp \leq n-1+p$$

Proof: Similar to that of Theorem 7.3 for min paths. //

Corollary 7.9

Let φ be a k out of n structure such that $\varphi \in \overline{\mathcal{S}}_p$ and $\varphi \notin \overline{\mathcal{S}}_{p+1}$. Then,

$$\frac{n-1}{p+1}+1 < k \leq \frac{n-1}{p}+1$$

Proof: By Theorem 7.8. //

Example 7.10

Can a 45 out of 65 structure support three statement AND logic? In other words is $\hat{C}_{\varphi 3}(A)=\emptyset$ for any A once the components are L-NAND consistent?

This requirement will be satisfied iff 45/65 belongs to $\mathcal{S}_4$. For this to be true k must satisfy the following,

$$k \geq \frac{(p-1)n+1}{p} = \frac{3n+1}{4} = 49$$

Therefore 45/65 does not belong to $\mathcal{S}_4$. Does it belong to $\mathcal{S}_3$? It does because k satisfies

$$k \geq \frac{2n+1}{3} = \frac{131}{3} \qquad \text{or if } k \geq 44.$$

Then the 45/65 structure belongs to S_3 and it can support up to two statement AND logic.

7.2 Restrictions on A and α to Assure Answers in Symmetric Structures

For any structure $\varphi \in \mathcal{S}_p$ with L-NAND consistent components, we know that it will be non contradictory ($\hat{C}_{\varphi q}(A) = \emptyset$ for all q, A) provided $C_{AND}(\mathcal{A}_\varphi(A)) \leq p-1$ or if $|\mathcal{A}_i| \leq p-1$ for all components $i=1,\ldots,n$. Further, $\check{C}_{\varphi q}(A) = \emptyset$ for all q and A when $\varphi \in \mathcal{S}_2$ with no further restriction on the complexity of statements. However, restrictions to assure $\hat{C}_{\varphi q}(A) = \emptyset$ do not guarantee that $\mathcal{A}_\varphi(A) \neq \emptyset$. In fact it is quite probable that $\mathcal{A}_\varphi(A) = \emptyset$ and we will have no passing statement. One way to ensure that $\mathcal{A}_\varphi(A) \neq \emptyset$ is to increase the number of alternatives (statements) $|\mathcal{A}_i(A)| = \alpha_i$ that each component i "accepts". In other words we require that $\alpha_i = \alpha$ for all components i. In the following we will determine how much α must be increased to ensure that $\mathcal{A}_\varphi(A) \neq \emptyset$.

Talking now of a structure $\varphi \in \bar{\mathcal{S}}_p$ with R-NOR consistent components we know that it will not be blocked. ($\check{B}_{\varphi q}(A) = \emptyset$ for all q and A). This means that it will not reject an identically true statement or more exactly, the disjunction of statements in $\bar{\mathcal{A}}_\varphi(A)$ will not produce an identically true statement as long as $C_{OR}(\mathcal{A}_\varphi(A)) \leq p-1$. This condition is satisfied when $|\bar{\mathcal{A}}_i(A)| \leq p-1$ for all components $i=1,\ldots,n$. For

simplicity we let $|\overline{\mathcal{A}}_i(A)|=\overline{\alpha}_i$ where $\overline{\alpha}_i=A-\alpha_i$ and where $A=|A|$. Similarly, $\overline{\alpha}=A-\alpha$ and again there is no guarantee that there will be answers (rejections in this case) since it is possible that $\overline{\mathcal{A}}_\varphi=\emptyset$. To make $\overline{\mathcal{A}}_\varphi\neq\emptyset$ we may increase $\overline{\alpha}$; but how much, will be examined in the following.

We will systematically talk about structures in $\mathcal{S}_p$. Results on $\overline{\mathcal{S}}_p$ follow directly by symmetrical arguments.

Before we proceed further we must resolve this question: Does it have meaning to impose restrictions on the number, α, of alternatives from A that each component is obliged to accept? When component i is forced to choose exactly α elements of A as passing ($|\mathcal{A}_i|=\alpha$) it is possible that, because of lack of choices in A, he may be forced to choose α alternatives that are i-false under AND conjunction, thus violating the component's L-NAND or NAND or R-NAND consistency. The same holds true for his rejection set $\overline{\mathcal{A}}_i(A)$ which will be required to satisfy $|\overline{\mathcal{A}}_i|=A-\alpha=\overline{\alpha}$, and again it is possible that he may choose statements whose OR disjunction is i-true thus violating the component's L-NOR or NOR or R-NOR consistency.

For example,

(a) If the component is L-NAND consistent (a requirement for $\varphi\in\mathcal{S}_p$ to be L-NAND consistent), we only need to be sure that there are α-plets of statements in A whose conjunc-

tion is not identically false.

(b) If the component is R-NOR consistent, (a requirement for $\varphi\in\bar{\mathcal{S}}_p$ to be R-NOR consistent) then we need to check if there are $\bar{\alpha}$-plets of statements in A whose disjunction is not identically true.

(c) If the component is NAND consistent (which is the same as NOR consistent) then we have to check that for each α-plet of statements in A whose conjunction is not i-false, the corresponding $\bar{\alpha}$-plet (of the rest of the statements) in A has a disjunction which is not i-true.

Otherwise the component is forced to violate his logic consistency "type". It is possible, especially in case (c) above, that there will be no α-plet ($\bar{\alpha}$-plet) to choose from. Take for example $A=\{a_1, a_2, a_3, \bar{a}_1, \bar{a}_2, \bar{a}_3\}$. Then if we restrict to $\alpha=2$ we will have $\bar{\alpha}$-plets that contain both a statement and its negation thus making their disjunction i-true.

This problem can be taken care of by allowing components to abstain on some statements. (Abstentions are studied in detail in Chapter 13.) If we study a structure $\varphi\in\mathcal{S}_p$ and we are interested in the L-NAND consistency of φ once the components are NAND consistent, then we only care that there are α-plets whose conjunctions are not i-false. So that the passing set of each component will contain an α-plet, since he is forced to accept one, without violating his NAND consistency. The possibility that his rejection set $\bar{\mathcal{A}}_i=A-\mathcal{A}_i$ may contain both a statement and its negation, is dealt with by declaring to components that such behaviour will be interpreted as abstention

from statements in $\bar{\mathcal{A}}_i(A)$.
The arguments are symmetrical when we focus instead on the rejection set $\bar{\mathcal{A}}(A)$ in case we deal with a structure $\varphi\in\bar{\mathcal{S}}_p$. In this case the existence of $\bar{\alpha}$-plets whose disjunction is not i-true is important. The passing set $\mathcal{A}_i(A)$ may be explained as implying abstention on some or all of its statements when the α-plet it contains is i-false under conjunction.

The method of increasing α is in fact a way to force the members of the social group to make compromises and accept more alternatives until a common agreement is formed. If an individual component or a social group in the structure is not able to make such compromises then he is given the opportunity to abstain from the decision structure.

Now back to our problem. We want to force the components to accept more statements in the hope that the structure will pass some statements. But how much force is enough?

Consider n components where each of them has $\alpha=A-1$. What is the k out of n structure with the longest k that we can construct and be certain that under all circumstances (any A) we will have $\mathcal{A}_\varphi\neq\emptyset$?

First we note that if $n\leq A-1$ then for any $k\leq n$ the k out of n structure, even the series structure n out of n, will have $\mathcal{A}_\varphi\neq\emptyset$ since each component can reject only one alternative

(α=A-1) and there are not enough components to reject all of the alternatives ($n \le A-1$).

In general, however, $n = A\rho + \epsilon$ where $A = |\mathcal{A}|$, ρ is a positive integer, $\epsilon \ge 0$ and integer with $\epsilon \le A-1$ and $1 \le \alpha \le A-1$

If we call

n_i: the number of components that pass statement a_i.

then we want to pick k so that

$$k \le \min_{\substack{\text{over all profiles of} \\ \text{voting situations}}} \left(\max_i (n_i) \right) \tag{7.7}$$

This way we are guaranteed that at least the issue that gets the max number of votes will pass the k out of n structure. The min in (7.7) is attained when passing votes are most "equally" divided among statements. When the total passing votes, αn, are "equally" divided among alternatives some alternatives will get $[\alpha n/A]$ and some will get $[\alpha n/A]+1$, where the symbol $[x]$ means "the integer part of x". In case $\alpha n/A$ is integer, all alternatives get $\alpha n/A$ votes. This implies that k must be chosen so that

$$k \le [[\alpha n/A]] \tag{7.8}$$

where the symbolism $[[x]]$ for $x \ge 0$ is defined as

$$[[x]] = \begin{cases} x & \text{if x is integer} \\ [x]+1 & \text{if x is not integer} \end{cases}$$

If each component i is obliged to pick α_i statements from

A then (7.8) is slightly generalized to,

$$K \leq [[\Sigma^{n}{}_{i=1}\alpha_i/A]] \qquad (7.9)$$

Using $n=A\rho+\epsilon$ (7.8) becomes,

$$K \leq [[A\alpha\rho/A+\alpha\epsilon/A]]=\rho\alpha+[[\alpha\epsilon/A]]$$

(7.10)

Technical reasons require the introduction of the quantities below and the presentation of their properties as they will serve later in the proofs of more interesting statements.

Let

$$\epsilon-\sigma=[[\epsilon\alpha/A]] \qquad (7.11)$$

but

$$[[\epsilon\alpha/A]] \leq \epsilon \qquad (7.12)$$

then

$$\sigma=\epsilon-[[\epsilon\alpha/A]]=[\epsilon-\epsilon\alpha/A]=[\overline{\alpha}\epsilon/A] \qquad (7.13)$$

Since

$$0 \leq \epsilon \leq A-1$$

$$0 \leq \sigma \leq [\overline{\alpha}\epsilon/A] \leq \overline{\alpha}-1 \qquad (7.14)$$

Also (7.10) which is the same as (7.8) can be written as,

$$K \leq \rho\alpha+\epsilon-\sigma \qquad (7.15)$$

Further we observe that an alternative way of expressing condition (7.8) is,

$$\frac{n}{n-K+1} < A/\overline{\alpha} \qquad (7.16)$$

as straightforward algebraic manipulation shows.

Lemma 7.11

The condition

$$\sigma A/\overline{\alpha} \le \epsilon < (\sigma+1) A/\overline{\alpha} \qquad (7.17)$$

is equivalent to the condition

$$\sigma = [\overline{\alpha}\epsilon/A]$$

Proof:

$\sigma = [\overline{\alpha}\epsilon/A] => \overline{\alpha}\epsilon/A = \sigma + \delta$ where $0 \le \delta < 1$

Then,

$$\epsilon = \sigma A/\overline{\alpha} + \delta A/\overline{\alpha}$$

and since $0 \le \delta < 1$, it follows that

$$\sigma A/\overline{\alpha} \le \epsilon < (\sigma+1) A/\overline{\alpha}$$

The steps are also reversible. //

Remark 1

Condition (7.9) is necessary and sufficient for a symmetric k out of n structure to have $\mathcal{A}_\varphi(A) \neq \emptyset$ because it represents the minimum as explained in (7.7)

Remark 2

The "dual" problem searches for the condition so that $\overline{\mathcal{A}}_\varphi \neq \emptyset$ for a k/n structure. This condition is

$$n-k+1 \le [[\frac{\Sigma\overline{\alpha}_i}{A}]] \qquad (7.18)$$

because there must be enough rejections so that even if they are equally divided among statements they exceed the length of a min cut (n-k+1).

Condition (7.18) can be rewritten as,

$$k \geq n-[[\frac{\Sigma\bar{\alpha}_i}{A}]]+1=n-n+[\frac{\Sigma\alpha_i}{A}]+1$$

because in general $[[J-b]]=J-[b]$ when J is positive integer and b some non negative number.

Hence,

$$k \geq [\frac{\Sigma\alpha_i}{A}]+1 \tag{7.19}$$

If we let $n=A\rho+\epsilon$ and $\alpha_i=\alpha$ for all i, then

$$k \geq \alpha\rho+[\frac{\epsilon\alpha}{A}]+1$$

Remark 3

Comparing (7.9) with (7.19) we see that unless $\Sigma\alpha_i/A$ is integer, choosing

$$k=[\frac{\Sigma\alpha_i}{A}]+1=[[\frac{\Sigma\alpha_i}{A}]]$$

both condition (7.9) and (7.19) are satisfied and the k/n structure will be guaranteed to have both $\mathcal{A}_\varphi \neq \emptyset$ and $\bar{\mathcal{A}}_\varphi \neq \emptyset$.

7.3 Answers with Logic for Certain Families of Symmetric Structures

In section 7.1 conditions were found for a k/n structure to belong to $\mathcal{S}_p$ (or $\overline{\mathcal{S}}_p$). Then using the results of Chapter 4, we know that by restricting $C_{AND}(\mathcal{A}_\varphi(A)) \leq p-1$, (or $C_{OR}(\overline{\mathcal{A}}_\varphi(A)) \leq p-1$), and components to be L-NAND (or R-NOR) consistent, the structure will respect logic i.e. $\hat{C}_{\varphi q}(A)=\emptyset$ for all q (or $\check{B}_{\varphi q}(A)=\emptyset$ for all q). This mechanism guarantees respect of logic but does not guarantee that there will be answers i.e. it is possible that $\mathcal{A}_\varphi=\emptyset$ (or $\overline{\mathcal{A}}_\varphi=\emptyset$). Consider for example, an 8 out of 10 structure. It will respect logic for level of complexity of conjunction of statements up to 7, but it most probably will be silent.

To have a non empty passing set (or rejection set) we have to impose condition (7.9) (or (7.19)). In this section, therefore, the compatibility of all these conditions is studied and families of k/n structures are determined for which they hold.

To connect with the famous "Possibility Theorem", we observe that Arrow's theorem proved that no coherent structure exists that will always give answers and respect logic, whether k/n or any other type, if α and A are not restricted. In our case, however, restrictions are imposed on α $(\alpha \leq p-1)$ and on each component being restrained to accept exactly α state-

ments from A. This in turn leads us to possible families of k/n structures.

In fact, if our search extends beyond the realm of k/n structures, which are attractive because of their symmetry (anonymity) and the operations that can be performed on them to reach other coherent structures, there will be wider families of coherent structures to satisfy the above conditions. But this will be further studied in Chapter 8.

The arguments below deal with strutures in $\mathcal{S}_p$ and not $\overline{\mathcal{S}}_p$ for reasons of economy. Similar results follow for $\overline{\mathcal{S}}_p$ by duality arguments and are presented in short in section 7.4.

Let us consider the conditions that must be imposed,

Condition 1 (Non emptiness of the passing set $\mathcal{A}_\varphi$)

$$k \le [[\Sigma^n{}_{i=1}\ \alpha_i/A]], \qquad \Sigma^n{}_{i=1}\ \alpha_i \ge 1 \qquad (7.20)$$

where $\alpha_i = |\mathcal{A}_i(A)|$

This Condition is necessary and sufficient to guarantee that $\mathcal{A}_\varphi \neq \emptyset$ (recall (7.9)).

When $\alpha_i = \alpha$ for all i, then Condition 1 becomes

$$k \le [[\alpha n/A]]\ , \qquad \alpha \ge 1 \qquad (7.21)$$

or

$$k \le \alpha\rho + [[\alpha\epsilon/A]] = \alpha\rho + \epsilon - \sigma \qquad (7.22)$$

(Recall (7.10), (7.11), (7.13), (7.14), (7.15) (7.16))

Condition 2 (φ belongs to $\mathcal{S}_p$)

$$pk \geq (p-1)n+1 \qquad (7.23)$$

Condition 2 is necessary and sufficient for a k/n structure to belong to $\mathcal{S}_p$ according to Theorem 7.3

Condition 3

(Limitation on the cardinality of the passing set of components)

$$\max_{i=1,\ldots,n} (\alpha_i) \leq p-1 \qquad (7.24)$$

Conditions 2 and 3 guarantee that $\hat{C}_{\varphi q}=\emptyset$ for all q (and $\check{C}_{\varphi q}=\emptyset$ for all q), In words, the outcome set $\mathcal{A}_\varphi$ will not give i-false statements under logic conjunction or disjunction of the statements in it, as long as the components are L-NAND consistent (recall Theorem 4.14, and Proposition 4.16)

In case $\alpha_i=\alpha$ for all components i then Condition 3 becomes,

$$\alpha \leq p-1$$

Condition 4 (Non triviality)

$$\max_i \alpha_i \geq 1 \quad \text{and} \quad \min_i \alpha_i \leq A-1 \qquad (7.25)$$

With this Condition the cases that all components have $\alpha_i=0$ or all have $\alpha_i=A$ are avoided. They are trivial anyway.

Once more if $\alpha_i=\alpha$ for all components i, then Condition 4 becomes,

$$1 \leq \alpha \leq A-1 \qquad (7.26)$$

In the following, except if explicitly stated, it will be assumed that $\alpha_i=\alpha$ for all components $i=1,\ldots,n$.

Theorem 7.12

Conditions 1, 2, 3, 4 are not compatible when φ is a k out of n symmetric structure and $\alpha_i=\alpha$ for all components i except when,

(a) $\alpha=p-1$, $A=p$, $n=p\rho+\epsilon$, $K=(p-1)\rho+\epsilon$ where $\rho\geq 0$ and integer and $1\leq\epsilon\leq p-1$ and ϵ is integer.

Or when,

(b) $0\leq K=n\leq A/(A-\alpha)$ where $\alpha\leq A-1$

Proof:

Let $n=A\rho+\epsilon$

Using Conditions 1 and 2 we obtain

$$(p-1)(A\rho+\epsilon)/p+1/p\leq K\leq\alpha\rho+\epsilon-\sigma \tag{7.27}$$

The proof now proceeds by examining cases:

Case 1:

Let $(p-1)/p<\alpha/A$ (7.28)

This Case is impossible since $\alpha\leq p-1$. We argue as follows,

(i) Let $\alpha=p-1$

Since $(p-1)/p<\alpha/A \Rightarrow A<p \Rightarrow A\leq p-1$.

But because of Condition 4 $A\geq\alpha+1=p$ and we are led to contradiction.

(ii) Let $\alpha=p-q$ $(q\geq 2)$, then α/A reaches its maximum value when A attains its minimum, namely when $A=p-q+1$, since $A\geq\alpha+1$. But then,

$$\frac{\alpha}{A}=\frac{p-q}{p-q+1}<\frac{p-1}{p} \qquad \text{for any } q\geq 2$$

Therefore, Case 1 is impossible.

Case 2:

Let $\alpha/A=(p-1)/p$

But we also demand that $\alpha\leq p-1$ and $A\geq\alpha+1$.

Now from $\alpha/A=(p-1)/p$ and $\alpha\leq p-1$ it follows that $A\leq p$.

However,

$\alpha/A=(p-1)/p$ and $A\geq\alpha+1$ imply that,

$\alpha p=A(p-1)\geq(\alpha+1)(p-1) \Rightarrow \alpha+1\geq p \Rightarrow A\geq\alpha+1\geq p \Rightarrow A\geq p$.

But both $A\leq p$ and $A\geq p$ imply $A=p$ and hence $\alpha=p-1$.

Substituting now in (7.27),

$$\frac{p-1}{p}(A\rho+\epsilon)+\frac{1}{p}\leq K\leq\alpha\rho+\epsilon-\sigma$$

where $0\leq\sigma\leq\overline{\alpha}-1 \Rightarrow \sigma=0$ (recall (7.15))

and $\sigma A/\overline{\alpha}\leq\epsilon<(\sigma+1)A/\overline{\alpha} \Rightarrow 0\leq\epsilon<A/\overline{\alpha}=p$ (7.29)

(recall (7.17)) we obtain,

$$\alpha\rho+(\epsilon\alpha/A)+(1/p)\leq\alpha\rho+\epsilon$$

and rearranging,

$$\epsilon\left(\frac{p-1}{p}-1\right)\leq-\frac{1}{p}$$

or $\epsilon/p\geq 1/p \Rightarrow \epsilon\geq 1$ (7.30)

Because of (7.29) though,

$0\leq\epsilon<A/\overline{\alpha}=A=p$ for the Case under examination. Hence,

$1 \le \epsilon < p$ (7.31)

We found then, that under the hypothesis of Case 2, Conditions 1, 2, 3, 4 are satisfied by the following k/n structures only

$$n = p\rho + \epsilon$$

$$(p-1)\rho + \epsilon(p-1)/p + 1/p \le k \le (p-1)\rho + \epsilon \qquad (7.32)$$

where $1 \le \epsilon \le p-1$, $\rho \ge 0$, $p \ge 2$, $A = p$, $\alpha = p-1$ and ϵ, ρ, p integers.

We can simplify (7.32) by noting that $(p-1)\rho+\epsilon$ is integer and that between it and the lower limit of k in (7.32) there is no other integer. To support this claim we will show that

$$(p-1)\rho+\epsilon-((p-1)\rho+\epsilon(p-1)/p+1/p) \ge 1$$

is impossible. Indeed, the above inequality leads to $\epsilon \ge p+1$ contradicting (7.32) which requires that $\epsilon \le p-1$. Consequently, (7.32) can be reformulated as,

$$n = p\rho + \epsilon$$

$$k = (p-1)\rho + \epsilon \qquad (7.33)$$

where, $\rho \ge 0$, $p \ge 2$, $1 \le \epsilon \le p-1$, $A = p$, $\alpha = p-1$ and ρ, ϵ, p integers.

And part (a) of the Theorem is proved.

Case 3:

Let $\alpha/A < (p-1)/p$

Note that since $\alpha \le p-1$ it follows that $A > p \Rightarrow A \ge p+1$

Also $\overline{\alpha} = A - \alpha \ge 2$ while from $\alpha/A < (p-1)/p \Rightarrow p > A/\overline{\alpha}$.

Now consider two subcases.

Case 3.1:

Let $\rho = 0$. Then $n = \epsilon$

Now condition (7.27) becomes,

$$\frac{(p-1)\epsilon+1}{p} \leq k \leq \epsilon-\sigma$$

from this we obtain

$$(p-1)\epsilon+1 \leq \epsilon p-\sigma p$$

or

$$\epsilon \geq 1+\sigma p \qquad (7.34)$$

Further by Condition 1 (recall (7.17)), we know that

$$\sigma A/\bar{\alpha} \leq \epsilon < (\sigma+1)A/\bar{\alpha}$$

From (7.34) and (7.17) it follows that,

$$\max(1+\sigma p,\ \sigma A/\bar{\alpha}) \leq \epsilon < (\sigma+1)A/\bar{\alpha}$$

which is equivalent to

$$1+\sigma p \leq \epsilon < (\sigma+1)A/\bar{\alpha} \qquad (7.35)$$

because

$$1+\sigma p > 1+\sigma A/\bar{\alpha} > \sigma A/\bar{\alpha}$$

From (7.35) it follows that,

$$1+\sigma p(A-\alpha) < (\sigma+1)A \Rightarrow A\sigma(p-1) < \alpha(1+\sigma p)$$

and rearranging terms,

$$\sigma(A(p-1)-\alpha p) < \alpha \quad \text{or}$$

$$\sigma\left(\frac{A}{\alpha}(p-1)-p\right) < 1 \qquad (7.36)$$

Look at the LHS of (7.36). As A/α increases the term in parentheses increases. What about σ? Recall that

$$\sigma = [\epsilon\bar{\alpha}/A]$$

therefore, as A/α increases, α/A decreases, $1-\alpha/A$ increases, $\bar{\alpha}/A$ increases and then σ increases as A/α increases.

Thus the LHS of (7.36) increases as A increases or α decreases. Since $A \geq p+1$ and $\alpha \leq p-1$, if we show that (7.36) does not hold for $A=p+1$ and $\alpha=p-1$, then it will not hold for any pair A, α so that $\alpha \leq p-1$ and $A \geq p+1$.

Substituting $A=p+1$ and $\alpha=p-1$ in (7.36) we obtain $\sigma<1$. But since σ is non negative integer, it follows that $\sigma=0$.

We conclude, therefore, that (7.36) does not hold for any $A \geq p+1$ and any $\alpha \leq p-1$ unless $\sigma=0$.

For $\sigma=0$ we have the following family of structures that satisfy Conditions 1, 2, 3, 4,

$$n=\epsilon$$

$$\frac{(p-1)\epsilon+1}{p} \leq k \leq \epsilon \tag{7.37}$$

with $0 \leq \epsilon < A/\overline{\alpha}$

We can simplify (7.37),

$((p-1)\epsilon+1)/p \leq \epsilon \Rightarrow (p-1)\epsilon+1 \leq \epsilon p \Rightarrow \epsilon \geq 1$

Also ϵ must satisfy,

$$0 \leq \epsilon < A/\overline{\alpha} \leq (p+1)/((p+1)-(p-1)) = (p+1)/2$$

Since the max value of $A/\overline{\alpha}$ is attained when $\overline{\alpha}/A$ is minimum or when α/A is maximum or when A/α is minimum; and this happens for $A=p+1$ and $\alpha=p-1$.

Finally, observe that between the numbers, $((p-1)\epsilon+1)/p$ and ϵ there is no integer other than ϵ.

This is verified by showing that

$$\epsilon-((p-1)\epsilon+1)/p \geq 1$$

is impossible.

Indeed, this inequality leads to $\epsilon \geq p+1$ and it contradicts

$\epsilon<(p+1)/2$ which was found above.

Therefore, (7.37) takes the simple form

$$1\leq k=n=\epsilon<A/\bar{\alpha} \qquad (7.38)$$

This family of structures is the family of series structures that does not have enough components to block all the statements in A. Note that the series structure belongs to $\mathcal{S}_p$ for all $p\geq 1$ and no restriction need be imposed on A and α apart from $n<A/\bar{\alpha}$.

This proves part (b) of the Theorem.

Case 3.2:

Let $\rho\geq 1$ then $n=A\rho+\epsilon$

Now (7.27) becomes,

$$A\rho(p-1)/p+(p-1)\epsilon/p+1/p\leq k\leq \alpha\rho+\epsilon-\sigma$$

or

$$(A(p-1)/p-\alpha)\rho\leq(\epsilon-1)/p-\sigma$$

Since $\rho\geq 1$ we obtain,

$$A(p-1)/p-\alpha+\sigma\leq(\epsilon-1)/p \qquad (7.39)$$

The RHS of (7.39) is not a function of A or α while the LHS is a function of both of them. In particular,

(i) The LHS increases as A increases, because the first term obviously increases in A while $\sigma=[\epsilon\bar{\alpha}/A]$ is increasing in A. The latter is true because $\bar{\alpha}/A=(A-\alpha)/A$ and it increases with A, approaching 1.

(ii) The LHS increases as α decreases because $\bar{\alpha}/A=$ $=(A-\alpha)/A$ increases as α decreases and therefore, $\sigma=[\epsilon\bar{\alpha}/A]$ increases as α decreases.

It follows now that if we show that (7.39) does not hold for the min values that A can take $(A\geq p+1)$ and the max value that α

takes ($\alpha \leq p-1$) then it will not hold for any pair α, A.

Therefore, substitute in (7.39) $\alpha=p-1$ and $A=p+1$ and aim towards contradiction,

$$\frac{(p+1)(p-1)}{p}-(p-1)\leq\frac{\epsilon-1}{p}-\sigma \quad =>$$

$=>\quad (p^2-1)/p-p+1\leq(\epsilon-1)/p-\sigma \quad =>$

$=>\quad (\sigma+1)p\leq\epsilon$

But $p>A/\overline{\alpha}$ by hypothesis of Case 3. Thus, $\epsilon>(\sigma+1)A/\overline{\alpha}$.

Contradiction, because by Condition 1 and (7.17) $\epsilon<(\sigma+1)A/\overline{\alpha}$. Therefore, Conditions 1, 2, 3, 4 are not compatible for Case 3.2 and the Theorem is proved. //

Example 7.13

(a) As an application to Theorem 7.12, construct a family of k/n structures that will always give answers ($\mathcal{A}_\varphi\neq\emptyset$) whose conjunction or disjunction will not be identically false or identically true respectively:

The set of statements contains 5 statements, $A=5$. Then $\alpha=A-1=4$ and $p=A=5$. We can choose any ϵ so that $1\leq\epsilon\leq p-1=4$.

Then choose $\epsilon=2$ and thus $n=p\rho+\epsilon=5\rho+2$ and $k=(p-1)\rho+\epsilon=4\rho+2$. The following table gives some of the k/n structure generated by these formulas:

ρ	0	1	2	3	4	5	100
n	2	7	12	17	22	27	502
k	2	6	10	14	18	22	402

To see how the 6/7 structure, for example, works to guarantee the non emptiness of the passing set $\mathcal{A}_\varphi$ when $\alpha=4$ and $A=5$ look at the 7 components. Each is obliged to reject exactly one of the statements in A. Since we deal with a symmetric structure if we try to block all issues the best we may hope for is that rejections are most equally divided among them, say,

comp. 1 rejects issue a_1
comp. 2 rejects a_2
comp. 3 rejects a_3
comp. 4 rejects a_4
comp. 5 rejects issue a_5
comp. 6 rejects a_1
comp. 7 rejects a_2

Therefore, three issues (a_3, a_4, a_5) get 6 passing votes each and two (a_1, a_2) get 5 votes each for a total of 7x4=28 passing votes. The three issues that get 6 votes each will pass through the 6/7 structure. If the votes are not as equally divided, then instead of three issues two or even one will pass but getting a greater number of votes.

Observe also that no more than three issues may pass and that as the structure belongs to $\mathcal{S}_5$ if the components are L-NAND consistent so is the structure.

(b) Consider the statements,

a_1: John has cold.

a_2: Whoever has cold coughs.

a_3: Whoever is near a coughing person gets cold.

a_4: I am near John.

a_5: I will not get cold.

This group of statements has the property that $(a_1 \wedge a_2 \wedge a_3 \wedge a_4) \Rightarrow$ NOTa_5. But any four of them are compatible under conjunction and hence L-NAND consistent components do not violate their logic consistency by choosing any four of them. The passing set of the 6/7 structure as described in (a) above has no problem either as it contains at most four elements anyway.

(c) Consider now the statements,

a_1: John has cold.

a_2: Whoever is cold is coughing or has fever or both.

a_3: John does not cough.

a_4: Whoever does not cough either does not have cold or is sleeping.

a_5: John is not cold.

the 4-plets (a_1, a_2, a_3, a_4) and (a_2, a_3, a_4, a_5) are the only 4-plets which are not i-false under conjunction. Therefore components in our 6/7 structure of (a) above can pick either. The outcome set is guaranteed to contain at least one statement but if it contains more, then it will not contain both a_1 and a_5 which are i-false under conjunction.

(d) Consider now a case from geometry,

a_1: The area of the triangle is 1/2.

a_2: The triangle is not isosceles.

a_3: The triangle is orthogonal.

a_4: The triangles longest side has length $\sqrt{2}$

a_5: One of the sides of the triangle has length 1.

First observe that 4-plets (a_2, a_3, a_4, a_5) and (a_5, a_1, a_2, a_3) are not possible choices for the components in the case of example in (a) above because each of them is i-false under conjunction of the statements it contains and the components are assumed L-NAND. The other three possible 4-plets are acceptable choices for the components. Observe though that the conjunction of the statements in any of those 4-plets implies the negation of the fifth statement: $(a_1 \wedge a_2 \wedge a_3 \wedge a_4) \Rightarrow NOTa_5$ $(a_3, a_4, a_5, a_1) \Rightarrow NOTa_2$ $(a_4, a_5, a_1, a_2) \Rightarrow NOTa_3$

Here the danger of getting contradictory answers in the passing set of the structure is obvious. By Theorem 7.12 we are guaranteed that we will get at least one statement in the outcome set and if there are more, their conjunction will not be i-false.

(e) An example of part (b) of Theorem 7.12 is easy to construct: Consider a 3 out of 3 structure with $A=7$ and $\bar{\alpha}=2$ then $k=n<A/\bar{\alpha}$. The conditions of the Theorem are satisfied and we expect at least one statement to pass. Indeed, if we try to reject the greatest possible number of issues the best we can do is,

comp. 1 rejects a_1 and a_2

comp. 2 rejects a_3 and a_4

comp. 3 rejects a_5 and a_6

Obviously issue a_7 remains not rejected and therefore passes the 3 out of 3 structure.

7.4 Duality

For structures in $\overline{S}_p$, Conditions 1, 2, 3, 4 take the following form when we replace k by n-k+1 and α by $\overline{\alpha}$.

Condition 1D

$k \geq [\Sigma\alpha_i/A]$ where $\Sigma\alpha_i \geq 1$

When $\alpha_i = \alpha$ for all $i = 1, \ldots, n$ then Condition 1 becomes

$k \geq [\alpha n/A] + 1$

which is equivalent to

$k \geq \alpha\rho + [\epsilon\alpha/A] + 1$ where $n = A\rho + \epsilon$

Condition 2D

$pk \leq n - 1 + p$

Condition 3D

$\max_i(\overline{\alpha}_i) \leq p - 1$

when $\overline{\alpha}_i = \overline{\alpha}$ for all components i, then this condition simplifies to

$\overline{\alpha} \leq p - 1$

Condition 4D

$\max_i(\overline{\alpha}_i) \geq 1$ and $\min_i(\overline{\alpha}_i) \leq A - 1$

when $\alpha_i = \alpha$ for all i then we simply require that

$1 \leq \overline{\alpha} \leq A - 1$

The dual to Theorem 7.12 is,

Theorem 7.14

Conditions 1D, 2D, 3D, 4D are not compatible when φ is restricted to the family of K out of n structures, except when:

(a) $\overline{\alpha}=p-1$, $A=p$, $n=p\rho+\epsilon$, $K=\rho+1$ where $\rho\geq 0$ and integer, $1\leq\epsilon\leq p-1$ and ϵ integer.

Or when,

(b) $K=1$, $n=A/\alpha$, $\alpha\leq A-1$

Proof:

Use dual arguments to those of Theorem 7.12 noting that when $K=(p-1)\rho+\epsilon$ the dual $n-K+1=\rho+1$ //

Part (b) of the Theorem represents a parallel structure that does not have enough components to pass all statements in *A*.

Example 7.15

Let the set of statements contain two elements, $A=2$. Then $\alpha=1=\overline{\alpha}$, $p=2$. Pick ϵ so that $1\leq\epsilon\leq p-1=1$. Then $\epsilon=1$.

Therefore the family of K/n structures of Theorem 7.14 is in this case given by,

$$n=p\rho+\epsilon=2\rho+1$$

$$K=(p-1)\rho+\epsilon=\rho+1$$

which represents the family of odd majority structures. Indeed, when $A=2$ in order to have answers that respect logic we force the components to choose one and only one statement and use an odd majority structure.

If instead we had $A=3$ then $\alpha=2$, $p=3$ and $1\leq\epsilon\leq 2$. This generates two families of K/n structures:

$n=3\rho+1$, $K=2\rho+1$: 3/4, 5/7, 7/10,.....

$n=3\rho+2$, $K=2\rho+2$: 4/5, 6/8, 8/11,.....

and components are obliged to pass two of the three issues in A to assure a non empty passing set whose elements are not i-false under AND conjunction.

Finally, if $A=3$ then $\bar{\alpha}=2$, $p=3$ and $0\leq\epsilon\leq2$. This generates the duals of the above families:

$n=3\rho+1$, $K=\rho+1$: 2/4, 3/7, 4/7,.....

$n=3\rho+2$, $K=\rho+1$: 2/5, 3/5, 4/7,.....

and components are forced to reject two of the three issues to guarantee a non empty rejection set whose elements do not form an i-true statement under disjunction.

Observe that the families for ρ large converge to 2/3 in the first case and 1/3 in the second ((p-1)/p and 1/p in general).

7.5 A Generalization : Requiring More Than One Answer

Condition 1 is required for the non emptiness of the passing set $\mathcal{A}_\varphi$ for k/n structures. A slight variation of Condition 1 will now be examined, that guarantees that $\mathcal{A}_\varphi$ contains at least λ elements ($\lambda \geq 1$). The compatibility requirement of this new condition as related to Conditions 2, 3, 4 will again lead to similar families of k/n structures.

Let once more,

α_i: the number of statements which component i is obliged to pass.

Then there are $\Sigma^n{}_{i=1}\alpha_i$ passing votes in total.

Let also,

n_j: the number of votes that the jth alternative in A attains.

Suppose, without loss of generality, that

$$n_1 \geq n_2 \geq n_3 \geq \ldots \geq n_A$$

Now if for example k is chosen so that $k \leq n_2$ then at least statements a_1 and a_2 will belong to $\mathcal{A}_\varphi$ (φ is k/n).

For reasons of clarity we start with a simple situation. We want to determine k so that the k/n structure will have at least two elements in $\mathcal{A}_\varphi$ under any voting situation i.e. no matter what values the n_j's may take.

The worst case (against our aim to have two elements in $\mathcal{A}_\varphi$) is when n_1 takes the largest possible value and n_2 takes the smallest possible value. This happens when,

$n_1 = n$ and $n_2 = [[(\Sigma\alpha_i - n)/(A-1)]]$

In words, when each component casts a vote for a_1, thus $n_1 = n$, and the rest of the issues get "equally" divided votes. In particular, n_2 gets an extra one vote if the division does not give an integer, because of the descending order in the n_i's.

Then the condition so that $|\mathcal{A}_\varphi| \geq 2$ becomes,

$$k \geq [[(\Sigma\alpha_i - n)/(A-1)]]$$

In general, we have,

Condition 1λ (For $|\mathcal{A}_\varphi| \geq \lambda$)

$$k \leq \left[\left[\frac{\sum_{i=1}^{n} \alpha_i - (\lambda-1)n}{A-(\lambda-1)}\right]\right] \qquad (7.40)$$

where $\Sigma\alpha_i \geq (\lambda-1)n+1$ (7.41)

When $\alpha_i = \alpha$ for all components i, then Condition 1λ becomes,

$$k \leq [[(\alpha-\lambda+1)n/(A-\lambda+1)]], \text{ with } \alpha-\lambda+1 \geq 1 \qquad (7.42)$$

Proposition 7.16

Condition 1λ is necessary and sufficient for a k/n structure to have $|\mathcal{A}_\varphi| \geq \lambda$.

Proof:

First note that the requirement (7.41) safeguards against the possibility that all votes go to λ-1 of the statements

(i) Sufficiency: When k satisfies (7.40) we know that $k \leq n_\lambda$ because the smallest possible value that n_λ can take under

any voting situation is when the first $\lambda-1$ statements get their largest value (total of $(\lambda-1)n$ votes allocated) and the rest of the votes $\Sigma\alpha_i-(\lambda-1)n$ are equally divided among the remaining $A-(\lambda-1)$ statements. Thus,

$$n_\lambda \geq \frac{\Sigma\alpha_i-(\lambda-1)n}{A-\lambda+1}$$

Therefore, since $k \leq n_\lambda$ at least λ of the statements get more or equal to k votes and thus pass the k/n structure.

(ii) Necessity: Suppose to the contrary that (7.40) does not hold. Then take the voting situation where the first $\lambda-1$ statements obtain n votes each and the rest of the votes be equally divided among the rest. Then,

$$n_\lambda \geq (\Sigma\alpha_i-(\lambda-1)n)/(A-\lambda+1)$$

but since $k>n_\lambda$ the λth statement will not pass the k/n structure thus $|\mathcal{A}_\varphi|=\lambda-1$. Contradiction. //

Based on Condition 1λ, Theorem 7.12 is now generalized,

Theorem 7.17

Conditions $1\lambda, 2, 3, 4$ are not compatible when φ is restricted to the family of k/n structures and $|\mathcal{A}_i|=\alpha$ for all components i, except when,

(a) $\alpha=p-1$, $A=p$, $n=(A-\lambda+1)\rho+\epsilon$, $k=(\alpha-\lambda+1)\rho+\epsilon$

where $\rho\geq 0$, $1+(\lambda-1)\rho\leq\epsilon\leq A-\lambda$ and ρ, ϵ, λ integers.

Or when

(b) $0\leq k=n\leq(A-\lambda+1)/\alpha$

Proof:

First observe that when $\varphi \in \mathcal{S}_p$ and $\alpha \le p-1$ then $|\mathcal{A}_\varphi| \le p-1$. Therefore, the requirement that $|\mathcal{A}_\varphi| \ge \lambda$ of Condition 1λ is consistent with Conditions 2, 3 only when $\lambda \le p-1$. This is indeed the case because by Condition 1λ we want $\lambda \le \alpha$ and by Condition 3 we want $\alpha \le p-1$.

Now proceed in parallel to the proof of Theorem 7.12 for which we had $\lambda = 1$.

Let $n = (A-\lambda+1)\rho+\epsilon \qquad 0 \le \epsilon \le A-\lambda$ and ϵ integer.

Then Condition 1λ becomes

$$k \le (\alpha-\lambda+1)+[[\frac{\epsilon(\alpha-\lambda+1)}{A-\lambda+1}]] \tag{7.43}$$

Let

$$[[\epsilon(\alpha-\lambda+1)/(A-\lambda+1)]] = \epsilon-\sigma \tag{7.44}$$

where $\sigma \ge 0$ and integer. $\sigma \ge 0$ because $\alpha-\lambda+1 \le A-\lambda+1$ $(\alpha \le A-1)$

Then

$$\sigma = [\epsilon(1-\frac{\alpha-\lambda+1}{A-\lambda+1})] = [\frac{\epsilon\overline{\alpha}}{A-\lambda+1}] \tag{7.48}$$

Then $0 \le \sigma \le \overline{\alpha}-1$ since $\epsilon \le A-\lambda$ and since $\overline{\alpha} \ge 1$ because $A-\alpha \ge 1$

Also

$\sigma = [\epsilon\overline{\alpha}/(A-\lambda+1)]$ is equivalent to

$$\sigma(A-\lambda+1)/\overline{\alpha} \le \epsilon < (\sigma+1)(A-\lambda+1)/\overline{\alpha} \tag{7.49}$$

Then Condition 1λ can be written as

$$k \le (\alpha-\lambda+1)\rho+\epsilon-\sigma \tag{7.50}$$

with $0 \le \epsilon \le A-\lambda$, $0 \le \sigma \le \overline{\alpha}-1$, $\sigma = [\epsilon\overline{\alpha}/(A-\lambda+1)]$, σ and ϵ integers.

Combining (7.50) with condition 2 we obtain,

$$\frac{p-1}{p}(A-\lambda+1)\rho+\epsilon\frac{p-1}{p}+\frac{1}{p}\le k\le(\alpha-\lambda+1)\rho+\epsilon-\sigma \qquad (7.51)$$

Case 1:

Let $(p-1)/p<\alpha/A$. This is impossible as in the proof of Theorem 7.12.

Case 2:

Let $(p-1)/p=\alpha/A \Rightarrow \alpha=p-1,\ A=p$

Then (7.51) becomes

$$\alpha(A-\lambda+1)\rho/A+\epsilon\alpha/A+1/A\le(\alpha-\lambda+1)\rho+\epsilon-\sigma \qquad (7.52)$$

Look at

$$\sigma=[\epsilon\overline{\alpha}/(A-\lambda+1)]$$

Since $\overline{\alpha}=A-\alpha=1$ and since $\sigma\le\overline{\alpha}-1 \Rightarrow \sigma=0$. Then (7.52) becomes

$$(\lambda-1)\rho+1\le\epsilon \qquad (7.53)$$

As long as (7.53) is satisfied, k can take any integer value in the interval defined by (7.51) or (7.52). But the interval contains only one integer: $(\alpha-\lambda+1)\rho+\epsilon$.

To show this we will prove that

$$(\alpha-\lambda+1)\rho+\epsilon-\alpha(A-\lambda+1)\rho/A-\epsilon\alpha/A-1/A\ge 1 \qquad (7.54)$$

is impossible. Manipulate (7.54) to obtain,

$$\rho(1-\lambda)(A-\alpha)+\epsilon(A-\alpha)-1\ge A$$

and since $A-\alpha=1$,

$$\epsilon\ge A+1+\rho(\lambda-1) \qquad (7.55)$$

But $\epsilon\le A-\lambda$ and $1\le\lambda\le p-1=\alpha$. Contradiction. Therefore,

(7.54) is not possible and hence,

$$K=(\alpha-\lambda+1)\rho+\epsilon \tag{7.56}$$

The only possible family of K/n structures that satisfy Conditions 1λ, 2, 3, 4 in Case 2 is given by

$$n=(A-\lambda+1)\rho+\epsilon$$

$$K=(\alpha-\lambda+1)\rho+\epsilon \tag{7.57}$$

where $1+(\lambda-1)\rho\le\epsilon\le A-\lambda$, $\alpha=p-1$, $A=p$, $\lambda\le\alpha$, and $\rho, \lambda, \epsilon, \alpha,$ p, A integers.

Case 3:

Let $\alpha/A<(p-1)/p$

Then $\alpha\le p-1$, $A\ge p+1 \Rightarrow \overline{\alpha}\ge 2$

Also $\alpha/A<(p-1)/p \Rightarrow p>A/\overline{\alpha}$

Case 3.1:

Let $\rho=0$ then $n=\epsilon$. Also (7.51) becomes,

$$\epsilon(p-1)/p+1/p\le K\le \epsilon-\sigma \tag{7.58}$$

and therefore $\epsilon\ge 1+p\sigma$. But

$$\sigma A/\overline{\alpha}\le\epsilon<(\sigma+1)(A-\lambda+1)/\overline{\alpha},$$

Therefore,

$$\max(1+\sigma p, \sigma A/\overline{\alpha})\le\epsilon<(\sigma+1)(A-\lambda+1)/\overline{\alpha}$$

but $1+\sigma p>1+\sigma A/\overline{\alpha}$ because $p>A/\overline{\alpha}$. Then,

$$1+\sigma p>1+\sigma(A-\lambda+1)/\overline{\alpha}$$

and therefore,

$$1+\sigma p\le\epsilon<(\sigma+1)(A-\lambda+1)/\overline{\alpha}$$

Further manipulation leads to,

$$\sigma((p-1)A/\alpha+(\lambda-1)/\alpha-p)+(\lambda-1)/\alpha<1 \tag{7.59}$$

In (7.59) observe that as A increases and/or α decreases, A/α increases. Also σ increases as A/α increases. And (λ-1)/α is nondecreasing as A increases and/or α

decreases.

If we show that when α takes its largest value ($\alpha=p-1$) and A its smallest ($A=p+1$) that (7.59) is impossible unless $\sigma=0$, then this will hold for any value of α, A. So let's substitute in (7.59) for $A=p+1$, $\alpha=p-1$

$$\sigma(1+(\lambda-1)/(p-1))+(\lambda-1)/(p-1)<1$$

If $\sigma\geq 1$ the inequality is violated and therefore $\sigma=0$.

Thus, we obtain the following family of k/n structures,

$$n=\epsilon$$

$$\frac{(p-1)\epsilon+1}{p}\leq k\leq\epsilon$$

with $0\leq\epsilon<(A-\lambda+1)/\overline{\alpha}$

But the interval within which k may vary contains only one integer, ϵ. Hence the family of structures is given by

$$0\leq k=\epsilon=n\leq(A-\lambda+1)/\overline{\alpha} \qquad (7.60)$$

This is a series structure, where the total rejections, $n\overline{\alpha}$, are not enough to block the $A-\lambda$ statements,

$$(A-\lambda+1\geq n\overline{\alpha}).$$

Case 3.2:

Let $\rho\geq 1$

This Case was proved to be impossible when $\lambda=1$ in the proof of Theorem 7.12. Condition 1λ becomes more limiting as λ increases while the rest of the Conditions 2, 3, 4 remain unchanged. Hence Conditions 1λ, 2, 3, 4 will remain incompatible for $\lambda\geq 1$. //

The dual to Condition 1λ is

Condition $1\lambda D$

$k\geq\alpha\rho+1+[\epsilon\alpha/(A-\lambda+1)]$

Also the "dual" to Theorem 7.17 is analogous to Theorem 7.14. Observe that for $\lambda=1$ we fall back to Condition 1D. Also note that for $\lambda\geq2$ there is no choice of k to guarantee both $|\mathcal{A}_\varphi|\geq\lambda$ and $|\overline{\mathcal{A}}_\varphi|\geq\lambda$ as it was possible in case when $\lambda=1$.

Discussion

If we are not interested in the logic implications of the statements that pass the structure, then we do not have to impose condition 3 ($\alpha\leq p-1$) or even Condition 2. Consider for example electoral votings where we are interested in the existence of one or more answers (The statements are of the form "I vote for G" or "I vote for L"). Then, of course any structure that satisfies Condition 1 and 4 is satisfactory.

If the possibility $\mathcal{A}_\varphi=\emptyset$ is allowed, but non contradictory answers are required we will keep Conditions 2, 3, 4 and relax Condition 1.

When the statements in A are statements of ordering objects, a wide range of literature has been developed studying possible restrictions of these orderings to ensure a passing set whose AND conjunction of statements (orderings) is not i-false (cyclic or non transitive).

Our approach on the other hand, dealing with statements in general, imposes restrictions on the form of the structure and the number of statements each component is obliged to accept as

related to the number of statements in the set of alternatives. In some sense the underlying ideas are similar. In both cases components are obliged to make compromises in their choices. The reward is of course a more logical common opinion. Afterall compromises is what any symbiosis is all about. However, if our feelings for more freedom do not allow us to reform our views we may abstain for a particular statement or totally in which case the structure with which decisions are taken will change as will be seen in more detail in the chapter on Abstentions.

One further remark is needed on "voting with weight". Instead of each component being obliged to choose α statements from A he can cast his α votes as he likes, to less than α issues or even all of them to one issue. This kind of behaviour has meaning in votings where logic is of no concern (electoral votings). This situation may take the usual forms we have dealt with by changing the symmetric structure k/n to $k/\alpha n$, where each component of the original structure is now split to α different components each accepting or rejecting, independently, exactly one statement from A.

Example 7.18

Let $A=6$, $\alpha=5$, $\lambda=2$. From Theorem 7.17,

$$1+(\lambda-1)\rho\leq\epsilon\leq A-\lambda$$

or

$$1+\rho\leq\epsilon\leq 4$$

and consequently ρ may vary from 0 to 3.

Further,

$$n=(A-\lambda-1)\rho+\epsilon=3\rho+\epsilon$$

$k=(\alpha-\lambda-1)\rho+\epsilon=2\rho+\epsilon$

The families of k/n structures, as defined by the above relations, are given in the following table,

ρ	0				1			2		3
ϵ	1	2	3	4	2	3	4	3	4	4
n	1	2	3	4	5	6	7	9	10	13
k	1	2	3	4	4	5	6	7	8	10

These are the families of k/n structures that will contain at least two statements in the passing set when A=6 and α=5, so that their AND conjunction does not produce an i-false statement.

Chapter 8

GROUP FORMATIONS WITHIN SYMMETRIC STRUCTURES: CONDITIONS FOR ANSWERS THAT OBEY LOGIC

8.1 Discussion, Motivating Example

The requirement for the existence of answers in k/n structures along with respect for logic lead us to very tight requirements on the acceptable families of k/n structures as was demonstrated in Chapter 7.

It now transpires that if components of a k/n structure form groups (parties) so that all the components in a group (party) express identical opinions then, depending on the multitude and size of the groups formed, the conditions on the form of the k/n structure, in order to safeguard for respect of logic and simultaneous existence of answers, will be relaxed and thus wider families of k/n structures than those determined in Chapter 7 will be accepted.

Naturally, the k/n structures along with the group (party) formations behave like a contraction of the original k/n structure (recall Chapters 5, 6). An example will help to present the problem with clarity.

Example 8.1

We are given a k/n structure with $k=19$ and $n=26$. Also $A=3$ and $\alpha=2$ while $p=3$ is required.

This structure does not belong to the families of structures found in Theorem 7.12. According to that Theorem, the family is given by $n=p\rho+\epsilon$, $k=(p-1)\rho+\epsilon$ and for $\rho=8$, $\epsilon=2$ the resulting compatible structure is $n=26$, $k=18$.

In our case, though, we have $k=19$, $n=26$. Our structure certainly

belongs to $\mathcal{S}_p$ ($p=3$) since the 18/26 structure does, but there is nothing to guarantee the existence of answers.

Assume now that the components of the 19/26 structure decide that giving up some of their traditionally respected individuality in order to form parties may not be that bad an idea after all. Immediatelly parties are formed and the structure is transformed; because these group formations are equivalent to the notion of contraction that was examined earlier.

In particular, let the following parties be formed:

G_1 contains 7 components

G_2 contains 5 components

G_3 contains 5 components

G_4 contains 5 components

G_5 contains 4 components

The groups are non overlapping and of course $\Sigma_i G_i=26$ where $G_i=|G_i|$.

Since there are 26 components and each makes two choices ($\alpha=2$) there will be 2x26=52 passing votes. Suppose that no groups are formed and that the three statements ($A=3$) receive most equally divided votes. The closest we can get to this is that two of them receive 17 votes each and one receives 18. (17+17+18=52 votes). This was expected since in the theory developed in Chapter 7 for $K\leq 18$ we expect at least one statement to pass the structure. But for $K=19$ this allocation of votes will result in an empty passing set.

In our case, though, we have the party formation $\mathbf{G}=(G_1,G_2,\ldots,G_5)$ in the 19/26 structure, the closest the three alternatives can get in the allocation of votes is,

a_1 gets 19 votes $(G_2+G_3+G_4+G_5)$

a_2 gets 17 votes $(G_1+G_2+G_4)$

a_3 gets 16 votes $(G_1+G_3+G_5)$

where $|G_i|=G_i$.

Therefore, the 19/26 structure will guarantee answers under any voting realization with $A=3$ and $\alpha=2$ (as well as the 18/26 structure). Both of them, of course, respect logic at level of complexity $p=3$.

The formation of groups, therefore, widened our choice introducing the 19/26 structure apart from the 18/26. In this Chapter we search for these wider families of structures that are permitted when groups or parties are formed.

8.2 Conditions for the Existence of Answers in Structures under Group Formations

Consider non overlapping groups of components $G_1, G_2, \ldots, G_g$ where each G_i contains G_i $(=|G_i|)$ components. The set A is as usual the set of statements and α is the number of statements each component is obliged to accept. What are the k/n structures $(n=\Sigma G_i)$ so that no matter how the g groups make their choices over A at least one statement passes the structure? We begin with some definitions:

Alternative a_i will get

$\Sigma^g{}_{j=1} q_{ij} G_j$ votes, where

$$q_{ij} = \begin{cases} 1 & \text{if group } G_j \text{ accepts } a_i \\ 0 & \text{if group } G_j \text{ does not accept } a_i \end{cases}$$

We want to look at the alternative that will get the maximum number of votes in any given voting matrix $\mathbf{Q}=\{q_{ij}\}$. Namely,

$$\max_i \sum_{j=1}^{g} q_{ij} G_j \qquad \text{for } i \in \{1, \ldots, A\}$$

Then we want to find the minimum of that maximum over all possible votings $\mathbf{Q}$.

This search is represented by the integer programming problem:

Problem Q

$$\text{Min } q$$

$$\mathbf{QG} \leq q\mathbf{e} \qquad (8.1)$$

$$\Sigma^{A}_{i=1} q_{ij} = \alpha \qquad \text{for } j=1,\ldots,g \qquad (8.2)$$

q_{ij} is a zero one variable and $q \geq 1$ and integer

where

$$\mathbf{G} = \begin{bmatrix} G_1 \\ . \\ . \\ G_g \end{bmatrix}$$

e is a vector of ones of length A and **q** is a Axg matrix of the variables q_{ij}.

Denote the optimal solution to Problem Q as $(\mathbf{Q}^*, q^*)$ which we will show below that it always exists. At optimality, at least one row of **QG** will be equal to q^*, otherwise q^* would not be optimal. Then any k/n structure with $k \leq q^*$ will guarantee answers, since no matter how the parties (groups) vote, when A and α are given, their will always be an alternative that receives at least q^* votes.

That condition $k \leq q^*$ is also necessary, is obvious. If to the contrary $k \geq q^*+1$, then there will be at least one voting situation, namely $\mathbf{Q}^*$, where all alternatives get less or equal than q^* votes and thus no alternative will pass the structure.

More formally,

Condition 1G (Existence of Answers)

$$k \leq q^* \tag{8.3}$$

where q^* is optimal for Problem Q.

Condition 1G is necessary and sufficient for the passing set of a k/n structure whose components have formed groups **G** ($\Sigma G_i = n$), to be non empty.

Lemma 8.1

Problem Q always has an optimal solution.

Proof:

First we show that a solution always exists:

Let G_1 vote for issues a_1 to a_α

G_2 vote for issues a_1 to a_α

. .

. .

G_g vote for issues a_1 to a_α

Then the first α rows of **QG** will each be less than or equal to $n(=\Sigma G_i)$ while the rest will equal zero. Thus a solution is $q=n$ and

$$q_{ij} = \begin{cases} 1 & \text{if } i \leq \alpha \text{ for all } j \\ \\ 0 & \text{if } i \geq \alpha+1 \text{ for all } j \end{cases}$$

Second observe that Problem Q is bounded since each q_{ij} is restricted to be 1 or 0. Therefore, an optimal solution always exists. //

Lemma 8.2

The optimal solution $\mathbf{Q}^*, q^*$ to Problem *Q* satisfies,

$$\alpha n \leq A q^* \tag{8.4}$$

where $n = \Sigma G_i$

Proof:

Take the sum of the inequalities of Problem *Q* at optimality.

//.

Inequality (8.4) was expected since the best we can hope to get is when all alternatives get equal number of votes, namely, $\alpha n/A$, if this quantity is integer.

Lemma 8.3

If $\alpha g \leq A$ then,

$$q^* = \max_i (G_i) \tag{8.5}$$

Proof:

Suppose there is a row whose RHS is greater than $\max_i G_i$, then q^* is not optimal as I can allocate one G_i to each row and attain $q^* = \max_i G_i$. Also I cannot get q^* less than $\max_i G_i$ since $G_{max} (= \max_i G_i)$ must vote anyway. //

Observe that in general,

$$q^* \geq \max_i (G_i) \tag{8.6}$$

Since the largest group will have to vote for some issue anyway.

Lemma 8.4

As long as $\alpha \leq A-1$, $\mathbf{Q}^*$ will have at least one zero in each row.

Proof:

Since there are Ag positions in the matrix $\mathbf{Q}$ and αg ones, there will be $(A-\alpha)g$ zeros in $\mathbf{Q}$. Since $\alpha \leq A-1$, there will be at least g zeroes in $\mathbf{Q}$ and therefore, in $\mathbf{Q}^*$ which are enough to allocate one to each row since $\mathbf{Q}$ is $A \times g$. If there is at least one zero in each row the optimal solution $q^*_1 < n = \Sigma G_i$; while if there is a row with all ones the optimal solution will be $q^*_2 = n = \Sigma G_i$ which is worse than q^*_1. Thus, $\mathbf{Q}^*$ will always have at least one zero in each row. //

Lemma 8.5

$$G_{max} \leq q^* \leq n - G_{min}$$

where $G_{max} \equiv \max_i (G_i)$, $G_{min} \equiv \min_i (G_i)$

Proof:

By Lemma 8.4 each row of $\mathbf{Q}^*$ has at least one zero, thus, $q^* \leq n - G_{min}$. Also $G_{max} \leq q^*$, since G_{max} has to vote for some issue. //

The optimal solution q^* can be thought of as a function of $\alpha, A, \mathbf{G}$. This can be expressed as,

$$q^* = q^*(\alpha, A, \mathbf{G})$$

Lemma 8.6

(i) $q^*(\alpha-1, A, \mathbf{G}) \leq q^*(\alpha, A, \mathbf{G})$ (8.8)

(ii) $q^*(\alpha, A+1, \mathbf{G}) \leq q^*(\alpha, A, \mathbf{G})$ (8.9)

Proof:

(i) Keep the row of $\mathbf{Q}^*(\alpha, A, \mathbf{G})$ that equals to $q^*(\alpha, A, \mathbf{G})$ intact. Then omit a one from each column of $\mathbf{Q}^*(\alpha, A, \mathbf{G})$ without touching the above row. This is possible since $\alpha \geq 2$ (otherwise $\alpha-1$ has no meaning) and therefore there are two ones in each

column of $\mathbf{Q}(\alpha, A, \mathbf{G})$. In this way a feasible solution to Problem Q with parameters $(\alpha-1, A, \mathbf{G})$ is obtained and such that $q(\alpha-1, A, \mathbf{G}) = q^*(\alpha, A, \mathbf{G})$. But then at optimality $q^*(\alpha-1, A, \mathbf{G}) \leq q(\alpha-1, A, \mathbf{G})$ and therefore $q^*(\alpha-1, A, \mathbf{G}) \leq q^*(\alpha, A, \mathbf{G})$.

(ii) Similarly. //

Definition

Let $\mathbf{G'} = (G_1, G_2, G_3, \ldots, G_g)$ and $\mathbf{G} = (G_1, G_2+G_3, \ldots, G_g)$ then we say that $\mathbf{G'}$ is *more refined* than $\mathbf{G}$

If $\mathbf{G'}$ is more refined than $\mathbf{G}$ and $\mathbf{G''}$ is more refined than $\mathbf{G'}$ then we will say again that $\mathbf{G''}$ is more refined than $\mathbf{G}$.

Lemma 8.7

Let $\mathbf{G'} = (G_1, G_2, G_3, \ldots, G_g)$ and $\mathbf{G} = (G_1, G_2+G_3, \ldots, G_g)$ i.e. $\mathbf{G'}$ is more refined than $\mathbf{G}$. Then,

$$q^*(\alpha, A, \mathbf{G'}) \leq q^*(\alpha, A, \mathbf{G}) \qquad (8.10)$$

Proof:

Take $\mathbf{Q}^*(\alpha, A, \mathbf{G})$ then construct the matrix $\mathbf{Q}(\alpha, A, \mathbf{G'})$ by splitting the column corresponding to G_2+G_3 in $\mathbf{Q}^*(\alpha, A, \mathbf{G})$ to two identical (one for G_2 and one for G_3). Then we obtain a solution to Problem Q with parameters $(\alpha, A, \mathbf{G'})$. Call the value of q corresponding to it $q_1(\alpha, A, \mathbf{G'})$. Certainly, $q_1(\alpha, A, \mathbf{G'}) = q^*(\alpha, A, \mathbf{G})$. But in general, $q_1(\alpha, A, \mathbf{G'}) \leq q^*(\alpha, A, \mathbf{G})$. Thus, $q^*(\alpha, A, \mathbf{G'}) \leq q^*(\alpha, A, \mathbf{G})$. //

In the limit case when $\mathbf{G}$ is refined so much that each G_i is split to its components (no formation of parties) we fall back to the theory on k/n structures with no group formations. In this case, $\mathbf{G}=\mathbf{e}_n=(1,1,\ldots,1)$ (vector of n ones) and

$$q^*(\alpha, A, \mathbf{e}_n)=[[n\alpha/A]] \tag{8.11}$$

as we have found from studying simple k/n structures.

Let us define the following problem

Problem *R*

$$\text{Max } r$$

$$\mathbf{RG}\geq r\mathbf{e} \tag{8.12}$$

$$\sum_{i=1}^{A} r_{ij}=A-\alpha \qquad \text{for } j=1,\ldots,g \tag{8.13}$$

where

$$r_{ij}=\begin{cases}1 & \text{if group j rejects issue } a_i\\ 0 & \text{if group j accepts issue } a_i\end{cases}$$

$r\geq 1$, $\mathbf{R}$ is the Axg matrix of r_{ij}'s (variables)

Lemma 8.8

Problem *Q* is equivalent to Problem *R*.

Proof:

Start with Problem *Q* and follow the series of transformations,

$$\text{Min } q \qquad\qquad \text{Max } n-q$$

$$QG \leq qe \qquad\qquad ne-QG \geq ne-qe$$

$$\sum_{i=1}^{A} q_{ij}=\alpha \qquad\qquad \sum_{i=1}^{A} q_{ij}=\alpha$$

$$\text{Max } n-q$$

$$\{[\mathbf{1}]-\mathbf{Q}\}\mathbf{G} \geq (n-q)\,\mathbf{e}$$

$$A-\sum_{i=1}^{A} q_{ij}=A-\alpha$$

where $[\mathbf{1}]$ is the Axg matrix whose elements are all ones. Now let,

$$r=n-q$$

$$\mathbf{R}=[\mathbf{1}]-\mathbf{Q} \tag{8.14}$$

then

$$\Sigma^{A}{}_{i=1} r_{ij}=A-\Sigma^{A}{}_{i=1} q_{ij} \tag{8.15}$$

and finally Problem *R* is obtained. //

It follows now that Condition 1G for the existence of answers, $k \leq q^*$ is equivalent to $k \leq n-r^*$.

Remark On Structure Duality

Consider the problem where each component rejects $\overline{\alpha}$ issues and we want to have at least one rejection by the structure (i.e. $\overline{\mathcal{A}}_{\varphi} \neq \emptyset$)

Again let

$$\bar{q}_{ij} = \begin{cases} 1 & \text{if issue } i \text{ is rejected by group } j \\ 0 & \text{otherwise} \end{cases}$$

then $\bar{\mathbf{Q}}$ is the Axg matrix of $\bar{q}_{ij}$'s with $\bar{\alpha}$ ones in each column.

Once more we search for

$$\underset{\bar{\mathbf{Q}}}{\text{Min}} \ (\text{Max}_i \ (\Sigma^{g}{}_{j=1}\bar{q}_{ij}) \ )$$

This is represented by the integer programming problem,

$$\text{Min } \bar{q}$$

$$\bar{\mathbf{Q}}\mathbf{G} \leq \bar{q}e$$

$$\Sigma^{A}{}_{i=1}\bar{q}_{ij} = \bar{\alpha}$$

But this is exactly Problem $Q(\bar{\alpha})$ where the parameter $\bar{\alpha}$ indicates that $\bar{\alpha}$ replaces α in the original Problem Q.

The corresponding condition for the non emptiness of the rejection set $(\bar{\mathcal{A}}_{\varphi} \neq \emptyset)$ is,

Condition 1G. D

$$n-K+1 \leq q^{*}(\bar{\alpha})$$

In a K/n structure n-K+1 is the length of a min cut and therefore following the same arguments as in the case of Condition 1G, we conclude that the condition is necessary and sufficient for a non empty rejection set $(\bar{\mathcal{A}}_{\varphi} \neq \emptyset)$.

It was shown that Problem $Q(\alpha)$ is equivalent to Problem $R(\bar{\alpha})$. It follows that Problem $Q(\bar{\alpha})$ is equiv-

alent to Problem $R(\alpha)$.

But comparing Problem $R(\alpha)$ with Problem $Q(\alpha)$ we see that at optimality

$$r^*(\alpha) \leq q^*(\alpha)$$

But $q^*(\bar{\alpha}) = n - r^*(\alpha)$ hence,

$r^*(\alpha) \leq q^*(\alpha) = n - r^*(\bar{\alpha})$ and therefore,

$r^*(\alpha) + r^*(\bar{\alpha}) \leq n$ which is equivalent to,

$$q^*(\alpha) + q^*(\bar{\alpha}) \geq n \tag{8.16}$$

Note now that the conditions :

for $\mathcal{A}_\varphi \neq \emptyset$ is $k \leq q^*(\alpha)$

for $\bar{\mathcal{A}}_\varphi \neq \emptyset$ is $n-k+1 \leq q^*(\bar{\alpha})$

Summing we obtain

$$n+1 \leq q^*(\alpha) + q^*(\bar{\alpha}) \tag{8.17}$$

which does not contradict (8.16). Therefore, the possibility that both $\mathcal{A}_\varphi$ and $\bar{\mathcal{A}}_\varphi$ will be non empty by appropriate choice of k is not excluded.

8.3 Conditions for the $\mathbf{G}$ Contraction of a Symmetric Structure to Belong to $\mathcal{S}_p$

Suppose that components within a symmetric structure form g parties, $\mathbf{G}=(G_1,\ldots,G_g)$, so that $\Sigma G_i=n$ where n as usual is the total number of components of the k/n symmetric structure. What is the condition that must be imposed on k of the k/n symmetric structure so that the contraction $K(\mathbf{G}|\varphi_{k/n})$ results in a structure that belongs to $\mathcal{S}_p$?

Certainly there is a number $k_o(\mathbf{G},p)$ so that when $k \geq k_o(\mathbf{G},p)$ then $K(\mathbf{G}|\varphi_{k/n}) \in \mathcal{S}_p$. The plan of attack in trying to determine $k_o(\mathbf{G},p)$ is roughly this:

Search over all possible k/n structures ($n=\Sigma G_i$) whose contraction according to $\mathbf{G}$ will result in at least one p-plet of min paths of the contracted structure not having a common component and therefore *not* belong to $\mathcal{S}_p$.

In this search of k/n structures we want to find the one with maximum k ($k_o(\mathbf{G},p)-1$) in the hope that when k is forced to be larger than that, ($k \geq k_o(\mathbf{G},p)$), all p-plets of min paths of the contracted structure will have at least one common component.

This problem is represented by what we will define as Problem *T*,

Problem *T*

Max t

so that

$\mathbf{TG} \geq t\mathbf{e}$ (8.1)

$\mathbf{T'G} \leq (t-1)\mathbf{e}$ (8.2)

There is at least one zero in each column (8.3)

There are no identical rows (8.4)

$t \geq 1$ and integer

where **T** is a pxg matrix composed of t_{ij}'s where

$$t_{ij} = \begin{cases} 1 & \text{if group } j \text{ belongs to path } i \\ 0 & \text{if group } j \text{ does not belong to path } i \end{cases}$$

T' is the missing element matrix of **T** (defined the same way $\mathbf{P}_\varphi'$ is defined w.r.t. $\mathbf{P}_\varphi$ in section 6.1)

Constraints (8.1) and (8.2) assure us that the p-plet of paths represented by **T** belong to the structure that results after contraction **G** is applied to the symmetric structure $t/\Sigma G_i$. To be more exact, the rows of a feasible solution **T** represent a p-plet of paths of a structure that results after contraction, K, w.r.t. **G** is applied to $t/\Sigma G_i$ and then omission of paths, OP,.
(For justification of this see Theorem 6.2)

Observe that constraint (8.2) also implies that no row of **T** is a strict subset of another. Here is a proof:

Let row r be strict subset of row 1. Then row r satisfies

$\Sigma^{g}{}_{j=1} t_{rj} G_j \geq t$

because of (8.35), but also is a subset of row 1 and therefore there is a row 1' of **T'** from those corresponding to row 1 of **T** so

that,

$$\sum_{j=1}^{g} t_{rj}G_j \le \sum_{j=1}^{g} t_{l',j}G_j \le t-1$$

and we have reached a contradiction.

Lemma 8.9

Let $g \ge p$. Then Problem T always has an optimal solution $\mathbf{T}^*, t^*$.

Proof:

Construct the following solution $\bar{\mathbf{T}}$, $\bar{t}$:

$$\bar{t}_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$$

$$\bar{t} = \min\{G_1, \ldots, G_p\}$$

Indeed, each inequality of $\bar{\mathbf{T}}\mathbf{G}$ consists of only one element (G_i) on the LHS and (8.1), (8.2) are satisfied. Also (8.3) and (8.4) are obviously satisfied and therefore, $\bar{\mathbf{T}}, \bar{t}$ is a feasible solution for Problem T. Further, since the t_{ij}'s are bounded, the problem is bounded and therefore it has an optimal solution, $\mathbf{T}^*, t^*$. //

Lemma 8.10

$$t^* \le n - G_{max}$$

where $G_{max} = \max_i (G_i)$

Proof:

t^* cannot be greater than $n-G_{max}$ because otherwise all rows will

be required to contain an element from G_{max} in the $t^*/\Sigma G_i$ symmetric structure, thus, violating constraint (8.3), since after contraction we obtain a one component cut structure.

//

Proposition 8.11

Let **G′** be more refined than **G**, then,

$$k_o(\mathbf{G}', p) \geq k_o(\mathbf{G}, p) \qquad (8.5)$$

(Note that $g' > g$ and $\Sigma^{g'}_{i=1} G'_i = \Sigma^{g}_{i=1} G_i = n$)

Proof:

If contraction according to the groups of **G′** is applied on $k_o(\mathbf{G}', p)/n$, a structure in $\mathcal{S}_p$ is obtained. If on this structure we further apply the contraction leading from **G′** to **G** (since **G′** is more refined than **G**), we will obtain a new structure in $\mathcal{S}_p$. But by the theory developed on contractions, we would obtain this final structure if we applied directly **G** on $k^o(\mathbf{G}', p)/n$. But $k_o(\mathbf{G}, p)$ is the smallest number so that when $k \geq k_o(\mathbf{G}, p)$, the contraction according to **G** applied to k/n leads to a structure in $\mathcal{S}_p$. Hence, $k_o(\mathbf{G}', p) \geq k_o(\mathbf{G}, p)$. //

Lemma 8.12

$$(p-1)n \geq t^* p \qquad (8.6)$$

whenever Problem T is feasible.

Proof:

Sum inequalities of (8.1) and observe that

$(p-1)\Sigma G_j \geq \Sigma_{i,j} t^*_{ij} G_j \geq pt^*$

since each G_j cannot appear more than p-1 times otherwise (8.3) is violated. //

Proposition 8.13

Whenever Problem T has a solution, $\mathbf{T}^*, t^*$, then the condition,

$$k \geq t^* + 1 \qquad (8.7)$$

is necessary and sufficient for the contraction of the symmetric structure $k/\Sigma G_j$ according to $\mathbf{G}$, to belong to $\mathcal{S}_p$.

Proof:

(a) Necessity: If it was not necesary, then a smaller k, say $k = t^*$ would be enough to guarantee that the contraction of $k/\Sigma G_i$ w.r.t. $\mathbf{G}$ belongs to $\mathcal{S}_p$. But the existence of an optimal solution t^* to Problem T contradicts this. Therefore, $k \geq t^*+1$ is necessary.

(b) Sufficiency: Apply contraction w.r.t. $\mathbf{G}$ on $t^*+1/\Sigma G_i$. According to the rules of contraction any p paths of the contracted structure will be non redundant thus satisfying constraints (8.1), (8.2), (8.4). But (8.3) will be violated (otherwise t^* would not be optimal). Thus, guaranteeing that any p paths will have at least one component in common. //

Proposition 8.14

If $g = p$, then a necessary and sufficient condition for the contraction of a $k/\Sigma G_i$ structure w.r.t. $\mathbf{G}$ to belong to $\mathcal{S}_p$ is that,

$$k \geq n - G_{max} + 1 \qquad (8.8)$$

Proof:

(a) Sufficiency: When $k \geq n - G_{max} + 1$ all paths of the k/n structure before contraction will contain at least one component from G_{max}. After contraction therefore, the resulting structure

will be a one component cut structure.

(b) Necessity: If $k \leq n - G_{max}$, then pick p min paths from the k/n structure as follows;

A min path that misses G_1 and perhaps others too.

A min path that misses G_2 and perhaps others too.

. . .

. . .

A min path that misses G_p and perhaps others too.

Such min paths exist in the k/n structure which has at least p=g min paths because there are at least g components ($n \geq g$) and since a k/n structure contains all possible combinations of k out of n components as min paths. But the above p-plet of min paths has no component in common after contraction w.r.t. **G**. Contradiction. //

Proposition 8.15

If $g < p$, a necessary and sufficient condition for the contraction $K(\mathbf{G} | (k/\Sigma G_i))$ to belong to $\mathcal{S}_p$ is that,

$$k \geq n - G_{max} + 1 \quad (8.9)$$

(this condition implies that the structure must be a one component cut structure)

Proof:

(a) Necessity.

Suppose to the contrary that $k \leq n - G_{max}$. For simplicity take $k = n - G_{max}$. Then in $\varphi = K(\mathbf{G} | (k/n))$ there will be a min path where G_{max} (or rather its contraction) is missing. Same is true for $G_1, \ldots, G_g$. Take all these paths of φ and say they are l in in number. Certainly $l \leq g$. These l min paths will have no component in common since for each one, one or more of the

G_j's is missing. Thus, we have found $1 \leq g < p$ min paths that have no component in common. Therefore, $\varphi \notin \mathcal{S}_p$. Contradiction.

(b) Sufficiency:

When $k \geq n - G_{max} + 1$ it follows that all paths of $k/\Sigma G_i$ contain at least one element from G_{max}. Thus, when contraction **G** is applied to k/n all paths will have a common component (being a one component cut structure). Therefore, the contracted structure belongs to $\mathcal{S}_p$, for any p. //

We can summarise Propositions 8.13, 8.14, 8.15 in the following Thorem,

Theorem 8.16

The following conditions are necessary and sufficient for the contraction a k/n symmetric structure w.r.t. **G** $(n = \Sigma G_i)$ to belong to $\mathcal{S}_p$:

(a) If $g \geq p$ that,

$$k \geq t^* + 1$$

where t^* is the optimal solution to Problem T.

(b) If $g < p$ that,

$$k \geq n - G_{max} + 1$$

where $G_{max} = \max_i (G_i)$.

(c) In particular when $p = g$ that,

$$k \geq n - G_{max} + 1 = t^* + 1 \quad \text{and} \quad t^* = n - G_{max}.$$

Proof:

By Propositions 8.13, 8.14, 8.15. //

8.4 A Heuristic Approach to Solving Problem *T*

When $g \geq p$ we may try to solve Problem *T* by first relaxing constraint 8.2 and then trying to see if small variations in the optimal solution of the relaxed problem make it feasible also for the original Problem *T*.

We will therefore examine the simpler,

Problem *U*

Max u

so that

$$\mathbf{UG} \geq u\mathbf{e} \qquad (8.10)$$

No row of **U** is redundant (8.11)

There is at least one zero in each column of **U** (8.12)

$u \geq 1$ and integer

where **U** is the pxg matrix of u_{ij}'s which are 0,1 variables and

$$\mathbf{G} = \begin{bmatrix} G_1 \\ \cdot \\ \cdot \\ G_g \end{bmatrix}$$

and **e** is a vector of ones of length p.

Constraint (8.10) guarantees that the minimum of the quantities

$$\sum_{j=1}^{g} u_{ij} G_j$$

over $i=1,\ldots,p$ is maximized over all possible u_{ij}'s

Constraint (8.11) says that no row of **U** is subset of another row. This implies that there is at least one zero in each row.

Constraint (8.12) does not allow the possibility that all p rows have a common component.

Before proceeding with the study of Problem *U* it is useful to define first the notion of *revolving min paths* or *revolving rows*.

Definition

Given g components and $g \geq p$, a set of p non redundant rows is called *revolving* iff each component appears in exactly p-1 of the rows and any p-1 rows do have at least one common component.

Let $P_1,\ldots,P_p$ be p revolving min paths. The components of p revolving paths can be divided into p non empty mutually exclusive and conclusive sets as follows:

$\mathfrak{C}_1$: set of components that are common among paths $\{P_2,P_3,\ldots,P_p\}$. $\mathfrak{C}_1$ is the intersection of these paths.

$\mathfrak{C}_2$: set of components that are common among paths $\{P_1,P_3,P_4,\ldots,P_p\}$

.

.

$\mathfrak{C}_p$: set of components that are common among paths $\{P_1,\ldots,P_{p-1}\}$

The $\mathfrak{C}_i$'s are mutually exclusive and conclusive. They are also

non empty, for otherwise, if, say, ϵ_i is empty then the p-1 rows: 1, 2, ..., i-1, i+1, ..., p would have no component in common which is a contradiction to the definition of revolving paths.

Now the paths will be:

$P_1=\epsilon_2 \cup \ldots \cup \epsilon_p$ (ϵ_1 is missing)

$P_2=\epsilon_1 \cup \epsilon_3 \cup \ldots \cup \epsilon_p$ (ϵ_2 is missing)

. . .

. . .

$P_p=\epsilon_1 \cup \epsilon_2 \cup \ldots \cup \epsilon_{p-1}$ (ϵ_p is missing)

which justifies the name *revolving paths*.

Lemma 8.17

Problem U always has an optimal solution when $g \geq p$.

Proof:

Take any p revolving paths. Then check that they satisfy all constraints. Further since u_{ij}'s are bounded (0, 1 variables), u is bounded and therefore, an optimal solution $\mathbf{U}^*, u^*$ always exists. //

Lemma 8.18

Any feasible solution to Problem U has at least one zero in each row.

Proof:

Otherwise all rows would be redundant in the presence of a row full of ones, thus, violating constraint (8.11). //

Lemma 8.19

Let $g \geq p$ and let an optimal solution be $\mathbf{U}^*, u^*$. Then there is an

optimal solution U^{**}, u^* in the form of revolving rows.

Proof:

The proof proceeds by constructing an optimal solution U^{**}, u^* in the form of revolving paths, starting from any optimal solution U^*, u^*. Assume sets $\epsilon_1, \ldots, \epsilon_p$ to be initially empty and assign columns (groups) to them as described below:

Look at the optimal solution U^*, u^*. Take the first row of U^*. It contains at least one zero say in the column j_1. Let component (column) j_1 belong to the set ϵ_1 which was initially empty.

Take the second row of U^*. It contains at least one zero say in column j_2. If $j_2 = j_1$ pick another column that has a zero in the second row. This is possible for otherwise row one is redundant in the presence of row two. Let component j_2 belong to ϵ_2 (which was initially empty).

Take the third row of U^*. Since rows 1, 2, 3 are not redundant among them, there is a $j_3 \neq j_1$ and $j_3 \neq j_2$ so that row 3 has a zero in column j_3. Again let j_3 belong to ϵ_3. Continue in this manner until row p and let j_p belong to set ϵ_p.

Now look at columns $\{1, 2, \ldots, g\} - \{j_1, j_2, \ldots, j_p\}$ (recall that by assumption $g \geq p$) and take them one by one: Pick column j and arbitrarily choose one of its zeros (it has at least one). Suppose the one we choose is in the ith row. Then j belongs to set ϵ_i. Continue in this way for each of the columns in $\{1, 2, \ldots, g\} - \{j_1, j_2, \ldots, j_p\}$.

The sets $\epsilon_1, \ldots, \epsilon_p$ constructed as described above are non empty and mutually exclusive, while $\epsilon_1 \cup \epsilon_2 \cup \ldots \cup \epsilon_p = \{1, 2, \ldots, g\}$.

Now for each ϵ_i change the columns of U^* belonging to it so that each column is composed of ones except for a zero in the ith

row. The new matrix will be now $\mathbf{U}^{**}$.

Constraint (8.10) will be satisfied since $u^{**}_{ij} \geq u^{*}_{ij}$ for all i, j. Therefore,

$$\mathbf{U}^{**}\mathbf{G} \geq u^{*}\mathbf{e}$$

Constraint (8.12) is satisfied since by construction $\mathbf{U}^{**}$ contains a zero in each column.

Finally the rows of $\mathbf{U}^{**}$ are not redundant as required by constraint (8.11): Pick any two rows r and t. Row r has a zero in column j_r (and perhaps other zeros also) and row t a zero in column j_t (and perhaps other zeros too) and $j_r \neq j_t$ (recall how $\{j_1, \ldots, j_p\}$ were picked). Thus, r and t are not subsets of each other.

It follows that $\mathbf{U}^{**}, u^{*}$ is an optimal solution since u^{*} was optimal by assumption.

By construction now, the first row of $\mathbf{U}^{**}$ is composed of components $\epsilon_2 \cup \epsilon_3 \cup \ldots \cup \epsilon_p$ (ϵ_1 is missing).

The second row is $\epsilon_1 \cup \epsilon_3 \cup \ldots \cup \epsilon_p$ (ϵ_2 is missing)

.   .

.   .

The pth row is $\epsilon_1 \cup \ldots \cup \epsilon_{p-1}$ (ϵ_p is missing)

Hence, $\mathbf{U}^{**}$ is formed of p revolving min paths. //

Lemma 8.20

If $g=p$ then $u^{*}=n-\max_j(G_j)$ (8.13)

Proof:

Using Lemma 8.19 we see that the p revolving paths of $\mathbf{U}^{*}$ will each contain all components except one. The smallest left hand side in the inequalities of constraint (8.10) will be the one for which the largest G_j is missing. Since Problem *U* maximizes

the minimum of the LHS of constraint (8.10), it follows that the minimum will be equal to u^*. Then $u^* = n - \max_j (G_j)$. //

Problem *U* can be shown to be equivalent to Problem *W* below when $g \geq p$,

Problem *W*

Min w

so that

$$\mathbf{WG} \leq w\mathbf{e} \tag{8.14}$$

$$\sum_{i=1}^{p} w_{ij} = 1 \qquad \text{for all } j=1,\ldots,g \tag{8.15}$$

where w_{ij}'s are 0,1 variables and $w \geq 0$ and integer while **W** is a pxg matrix of w_{ij}'s.

Intuitively, Problem *U* searches over possible revolving paths. That is it searches over possible p sets of G_i's: $E_1,\ldots,E_p$ (where E_i is defined as the union of the G_j's belonging to ϵ_i for each $i=1,\ldots,p$) and seeks to maximize the minimum (w.r.t. cardinality: number of components it contains), union of any p-1 of the E_i's. This is the same as minimizing the maximum E_i by looking at all possible formations $E_1,\ldots,E_p$. This is exactly what Problem *W* does.

More formally Problem *U* can be transformed to Problem *W* in the following way:

When $g \geq p$ it was shown that there is always an optimal solution in the form of revolving paths. Therefore, Problem *U* can be written as:

Max u Min $n-u$

so that   so that

$\mathbf{UG} \geq u\mathbf{e}$ $n\mathbf{e}-\mathbf{UG} \leq (n-u)\mathbf{e}$

$\sum_{i=1}^{p} u_{ij} = p-1$ $p-\sum_{i=1}^{p} u_{ij} = 1$

$\mathbf{U}$ has the form of $\mathbf{U}$ has the form of

revolving paths   revolving paths

Let $w=n-u$ and $w_{ij}=1-u_{ij}$ for all i, j. Let $[1]$ be the pxg matrix whose elements are all ones and note that $[1]\mathbf{G}=n\mathbf{e}$. Then

$n\mathbf{e}-\mathbf{UG}=([1]-\mathbf{U})\mathbf{G}=\mathbf{WG}$. Also,

$p-\Sigma^{p}_{i=1}u_{ij}=\Sigma^{p}_{i=1}(1-u_{ij})=\Sigma^{p}_{i=1}w_{ij}$

Thus we obtain,

Min w

$\mathbf{WG} \leq w\mathbf{e}$

$\Sigma^{p}_{i=1}w_{ij}=1$

$[1]-\mathbf{W}$ has the form of revolving paths

The constraint that $[1]-\mathbf{W}$ has the form of p revolving paths is not necessary since without it:

(a) Each column has at least one one because of $\Sigma_i w_{ij}=1$ and hence each G_i belongs to some row of $\mathbf{W}^*$.

(b) There is an optimal solution so that no row of $\mathbf{W}^*$ is all zeros; for if this were true some other row would contain at least two of the G_i's (since $g \geq p$). But then the optimality of $\mathbf{W}^*$ is not mared by transfering one G_i from this row that contains at least two to the one that contains none.

We reach therefore,

Problem W

Min w

so that

$\mathbf{WG} \leq w\mathbf{e}$

$\Sigma^{p}_{i=1} w_{ij} = 1$

$w \geq 1$ integer and w_{ij} 's are 0, 1 variables

where we assume that the optimal solution chosen will be one that has no row whose elements are all zero as explained in (b) above.

It is certainly more convenient to deal with Problem W and then return to Problem U where $u^* = n - w^*$ and $\mathbf{U}^* = [\mathbf{1}] - \mathbf{W}^*$, where $\mathbf{W}^*, w^*$ is an optimal solution to Problem W.

Now let

$$\epsilon_i \equiv \{ j \mid w^*_{ij} = 1 \} \qquad (8.16)$$

ϵ_i is the set of indices corresponding to those columns of $\mathbf{W}^*$ where a one appears in row i (i.e. it represents the non zero elements of row i of $\mathbf{W}^*$ and therefore is min path i of the contraction $\mathbf{G}$ on some symmetric structure with $n = \Sigma G_i$ components)

Certainly, $\epsilon_i \cap \epsilon_j = \emptyset$, $\epsilon_i \neq \emptyset$ for all i,

$\epsilon_1 \cup \epsilon_2 \cup \ldots \cup \epsilon_p = \{1, \ldots, g\}$. Also let,

$$E_i = \bigcup_{j \in \epsilon_i} G_j \qquad i = 1, \ldots, p \qquad (8.17)$$

and again $E_i \neq \emptyset$ for all i, $E_i \cap E_j = \emptyset$,

$E_1 \cup \ldots \cup E_p = \cup^{g}_{i=1} G_i = \{1, \ldots, n\}$.

Finally let,

$$e_i \equiv |E_i| = \sum_{j \in \mathcal{E}_i} G_j = \sum_{j=1}^{g} w^*_{ij} G_j \qquad (8.18)$$

Therefore, the p revolving paths of $\mathbf{U}^*$ will be given by $\mathbf{U}^* = [\mathbf{1}] - \mathbf{W}^*$ or more specifically,

row $i: \{1, \ldots, g\} - \mathcal{E}_i = \mathcal{E}_1 \cup \ldots \cup \mathcal{E}_{i-1} \cup \mathcal{E}_{i+1} \cup \ldots \cup \mathcal{E}_p$

for $i = 1, \ldots, p$ which has the form of revolving paths.

Observe that by taking the union of any p-1 of the E_i's we can form revolving paths as well, for some structure that contains $n = \Sigma G_i$ components. Intuitively, the $\mathcal{E}_i$'s refer to the contracted structure with g components, and the E_i's refer to the structure before contraction with $n = \Sigma G_i$ components.

Problem W can be given a simple physical meaning: We are given g sticks each with length G_i, $i = 1, \ldots, g$, and we want to divide them among p persons, $g \geq p$, (person i gets sticks whose indices are in set $\mathcal{E}_i$) so that the total length, $|E_i| = e_i$, that each person i receives are "most equal". Where by "most equal" we mean that we will minimize the maximum length that any person may receive. For this reason we will also refer to Problem W as the "Sticks Allocation Problem".

Some properties of u^* and w^* are now presented.

Lemma 8.21

$$(p-1)n \geq pu^* \qquad (8.19)$$

Proof:

Sum the inequalities of Problem U at optimality using the fact that the optimal solution has the form of revolving paths and therefore has p-1 ones in each column of $\mathbf{U}^*$. //

Certainly u^* and w^* are functions of p and $\mathbf{G}$,

$$u^*=u^*(p,\mathbf{G}) \quad \text{and} \quad w^*=w^*(p,\mathbf{G})$$

Lemma 8.22

$$u^*(p+1,\mathbf{G}) \geq u^*(p,\mathbf{G}) \tag{8.20}$$

$$w^*(p+1,\mathbf{G}) \leq w^*(p,\mathbf{G}) \tag{8.21}$$

as long as $g \geq p+1$

Proof:

Take $\mathbf{W}^*(p,\mathbf{G})$, the optimal matrix for parameters $p,\mathbf{G}$. Construct $\mathbf{W}(p+1,\mathbf{G})$ by adding a row to $\mathbf{W}^*(p,\mathbf{G})$ and transfering to it ones from other rows that have more than one ones. There is at least one row that has at least two ones since $g \geq p+1$. The resulting matrix $\mathbf{W}(p+1,\mathbf{G})$ satisfies,

$$\mathbf{W}(p+1,\mathbf{G})\mathbf{G} \leq \mathbf{W}^*(p,\mathbf{G})\mathbf{G}$$

Therefore at optimality

$$w^*(p+1,\mathbf{G}) \leq w^*(p,\mathbf{G})$$

Also since $w^*=n-u^*$, it follows that

$$u^*(p+1,\mathbf{G}) \geq u^*(p,\mathbf{G}) \qquad //$$

Lemma 8.23

Let $\mathbf{G}'$ be more refined than $\mathbf{G}$ then,

$$w^*(p,\mathbf{G}') \leq w^*(p,\mathbf{G})$$

(or $u^*(p,\mathbf{G}') \geq u^*(p,\mathbf{G})$),

Proof:

Take $\mathbf{W}^*(p, \mathbf{G})$ and split the columns of those G_i;s that are more refined in $\mathbf{G}'$, in identical columns. The resulting $\mathbf{W}(p, \mathbf{G}')$ satisfies

$$\mathbf{W}(p, \mathbf{G}')\mathbf{G}' \leq w^*(p, \mathbf{G})$$

Therefore at optimality the maximal element of $\mathbf{W}^*(p, \mathbf{G}')\mathbf{G}'$ will still be not greater than $w^*(p, \mathbf{G})$. Thus,

$$w^*(p, \mathbf{G}') \leq w^*(p, \mathbf{G}) \qquad //$$

In the limit case where the G_i's are so refined so that $G_i=1$ for all i, we are back at the simple k/n structure situation

$$w^*(p, \mathbf{1}) \leq w^*(p, \mathbf{G}) \quad \text{for all } \mathbf{G}$$

where $\mathbf{1}$ is a vector of ΣG_i ones.

In fact, recalling that Problem *W* with parameters $(p, \mathbf{1})$ divided n units most equal among p persons we conclude that

$$w^*(p, \mathbf{1}) = [[n/p]]$$

Therefore,

$$u^* = n - w^* = n - [[n/p]] = [n - n/p] = [n(p-1)/p]$$

which reminds us of the condition

$$k \geq (n(p-1)+1)/p$$

which is the same as $k \geq u^*+1$ since k must be integer and u^* is integer.

Example 8.24

Let $G_1=5$, $G_2=3$, $G_3=3$, $G_4=2$, $G_5=1$, $G_6=2$, and $p=3$. Then $g=6$ and $g \geq p$. To solve Problem *W* allocate the G_i's to E_1, E_2, E_3 as follows,

$E_1 = G_1 \cup G_5 \Rightarrow |E_1| = 5+1 = 6$ components

$E_2 = G_2 \cup G_6 \Rightarrow |E_2| = 3+2 = 5$ components

$E_3 = G_3 \cup G_4 \Rightarrow |E_3| = 2+3 = 5$ components

Then $w^*=6$ since the above allocation minimizes the maximum $|E_i|$ and is therefore optimal.

$$W^* = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

$$\mathbf{U}^* = [\mathbf{1}] - \mathbf{W}^* = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

$u^*=n-w^*=\Sigma G_i-w^*=16-6=10$

The p revolving paths of $\mathbf{U}^*$ are:

Path 1: $E_2 \cup E_3$ (it contains 10 components)

Path 2: $E_2 \cup E_3$ (it contains 11 components)

Path 3: $E_1 \cup E_3$ (it contains 11 components)

Check now that constraint (8.2) of Problem T is not satisfied. However, changing $\mathbf{U}^*$ by setting all elements in the 5th column equal to 0, the optimal solution $u^*=10$ does not change and the constraints of Problem Y are now satisfied. Therefore, $t^*=u^*$.

If $k \geq u^*+1=10+1=11$ then any k/16 structure, contracted according to **G** will belong to $\mathcal{S}_p = \mathcal{S}_3$.

If there was no contraction the condition would be

$k \geq ((p-1)n+1)/p=11$

which in this case is the same as when we have contraction.

Example 8.25

Let $G_1=7$, $G_2=6$, $G_3=6$, $G_4=3$ and $p=3$. Solving Problem W we

allocate:

$E_1=G_1$, $E_2=G_2$, $E_3=G_3 \cup G_4$ and hence,

$e_1=7$, $e_2=6$, $e_3=9$ and $w^*=9$. Then $u^*=n-w^*=22-9=13$.

The rvolving paths are,

$E_2 \cup E_3$ containing $6+(6+3)=15$ components$=12+3$

$E_1 \cup E_3$ containing $7+(6+3)=16$ components $=13+3$

$E_1 \cup E_2$ containing $7+6=13$ components

These paths do not satisfy constraint (8.2) of Problem T (the second path violates it). But by altering the path so that it contains G_1, G_3 but not G_4, we see that u^* does not change and that now it satisfies the constraints of Problem T. Therefore, $t^*=u^*=13$ and the requirement that $k \geq t^*+1=14$ guarantees that any k/22 structure, when contracted according to **G**, will belong to $\mathcal{S}_3$.

When there is no contraction in the k/22 structure then the requirement that it belongs to $\mathcal{S}_3$ is equivalent to the condition found for k/n structures, $k \geq ((p-1)n+1)/p=(2*22+1)/3=15$, which is an upper bound for t^*+1 for any contraction **G**.

Example 8.26

Let the G_i's as in Example 8.25 but now $p=4$. Then $p=g$. Problem W will allocate: $E_1=G_1$, $E_2=G_2$, $E_3=G_3$, $E_4=G_4$. Since $G_1=G_{max}=7$, then $w^*=G_{max}=7$ and $u^*=n-w^*=$ $=22-7=15$. To satisfy Condition 2G we want $k \geq n-G_{max}+1=t^*+1=$ $=u^*+1=16$.

Example 8.27

Suppose that $G_1=3$, $G_2=2$, $G_3=1$ and $p=5$. In this case $g=3$ and $g<p$.

It is not possible to construct 5 non redundant min paths. We construct instead a one component cut structure (after contraction) by picking $k \geq n-G_{max}+1$ or $k \geq 6-3+1=4$. Recall that a one component cut structure belongs to $\mathcal{S}_p$ for any p by definition.

8.5 On the Compatibility of Conditions for Respect of Logic and Existence of Answers under Group Formations (Contractions) in Symmetric Structures.

Families of symmetric structures are found, that under a given group formation $\mathbf{G}=(G_1,\ldots,G_g)$ within them, along with restrictions on α, A, p satisfy both Conditions 1G and 2G as defined previously.

The Conditions whose compatibility is to be examined are,

Condition 1G (existence of answers)

-If $\alpha g \geq A$ then $k \leq q^*$ where q^* is the optimal solution for Problem *Q*.

-If $\alpha g \leq A$ then $k \leq \max_i(G_i)$.

Condition 2G ($\varphi \in \mathcal{S}_p$)

-If $g \geq p$ then $k \geq t^*+1$ where t^* is optimal for Problem *T*.

-If $g \leq p$ then $k \geq n-\max_i(G_i)+1$ where $n=\Sigma G_i$

Condition 3 (respect for p-level logic)

$\alpha \leq p-1$

Condition 4 (non triviality)

$1 \leq \alpha \leq A-1$

Conditions 1G and 2G have been extensively discussed in the two previous sections. Conditions 3 and 4 are familiar from Chapter 7.

To study the compatibility of the above conditions we follow a

line of thought parallel to that of Chapter 7 by examining cases. The results of this investigation are summarized at the end in Theorem 8.28.

Case 1:

$$\frac{\alpha}{A} > \frac{p-1}{p}$$

This Case is impossible because of Condition 3. A proof appears in Chapter 7 (7.28).

Case 2:

$\alpha/A = (p-1)/p \iff \alpha = p-1$, $A = p$ because of Condition 3.

Case 2.1:

Let $\alpha g \geq A$ and $g \leq p$ then $p \geq g \geq A/\alpha = p/(p-1)$ which holds for $p \geq 2$.

Conditions 1G and 2G require that,

$$n - G_{max} + 1 \leq k \leq q^*$$

or that

$$n - G_{max} + 1 \leq q^* \tag{8.22}$$

Is there some condition to guarantee (8.22)? In fact we will show there is one.

First observe that

$$G_{max} \geq n/g$$

because the maximum group is not less than the average ($\Sigma G_i = n$).

But $g \leq p \Rightarrow n/g \geq n/p$. Therefore,

$$n - G_{max} \leq n - n/p = (p-1)n/p \tag{8.23}$$

Also, we know that

$$\alpha n/A \leq q^* \tag{8.24}$$

Therefore,

$$n-G_{max} \leq (p-1)n/p \leq q^* \tag{8.25}$$

If $(p-1)n/p$ is not integer and since $n-G_{max}$ and q^* are integers we may conclude from (8.67) that

$$n-G_{max}+1 \leq [[(p-1)n/p]] \leq q^* \tag{8.26}$$

Therefore, whenever $(p-1)n/p$ is not integer (8.26) holds, which implies the truth of (8.22) and hence there is a k/n structure which after contraction according to **G** respects logic and gives answers. Observe that the contracted structure is a one component cut structure anyway because of (8.22).

Case 2.2:

Let $\alpha g \geq A$ and $g \geq p$, then $g \geq \max\{A/\alpha,\ p\} = \max\{p/(p-1),\ p\} = p$

Now Conditions 1G and 2G require that

$$t^*+1 \leq k \leq q^*$$

On the other hand, we know that

$$t^* \leq (p-1)n/p \text{ and } \alpha n/A \leq q^* \text{ or } (p-1)n/p \leq q^*$$

hence,

$$t^* \leq (p-1)n/p \leq q^*$$

This condition implies that $t^*+1 \leq q^*$ as long as $(p-1)n/p$ is not integer.

Case 2.3:

Let $\alpha g \leq A$ and $g \leq p$ then,

$$g \leq \min\{A/\alpha, p\} = \min\{p/(p-1),\ p\} = p/(p-1)$$

Therefore, $g=1$ for $p \geq 3$ and $g=2$ for $p=2$.

Now Conditions 1G, 2G become,

$$n-G_{max}+1 \leq k \leq G_{max} \tag{8.27}$$

It follows that,

$$n-G_{max}+1 \leq G_{max} \quad \text{or}$$

$$G_{max} \geq (n+1)/2 \qquad (8.28)$$

which is necessary and sufficient for (8.27)

Observe that $k \leq G_{max}$ implies the existence of a one component path (which results from the contraction of G_{max}). Also, the condition $k \geq n-G_{max}+1$ implies the existence of a one component cut (G_{max} after contraction). Therefore, G_{max} is a dictatorial group that controls both a path and a cut and reduces to a single component after contraction. This was expected since $g=2$ and thus only two groups exist. In order to have compatibility of Conditions, one group must become irrelevant after contraction.

Case 2.4:

Let $\alpha g \leq A$ and $g \geq p$ then $p \leq g \leq A/\alpha$

Conditions 1G, 2G require that

$$t^*+1 \leq k \leq G_{max}=q^*$$

But we know that

$$t^* \leq (p-1)n/p=\alpha n/A \leq q^*=G_{max} \qquad (8.29)$$

Then as long as $(p-1)n/p$ is not integer (8.29) implies

$$t^*+1 \leq [[(p-1)n/p]] \leq q^*=G_{max} \qquad (8.30)$$

which is sufficient for the compatibility of conditions 1G, 2G. Note that in this Case, $k \leq G_{max}$ implies that after contraction there is a one component path and hence answers are guaranteed under all circumstances.

Case 3:

$$\alpha/A<(p-1)/p$$

Cases 3.1, 3.2, 3.3:

These Cases are as in Cases 2.1, 2.2, 2.3 respectively. But now a simple condition for compatibility cannot be derived. Depending on each particular case, Conditions 1G, 2G, 3, 4 may or may not be compatible.

Case 3.4:

Let $\alpha g \leq A$ and $g \leq p$ then $g \leq \min\{A/\alpha, p\}$.

Again a necessary and sufficient condition $k \leq G_{max}$ and $k \geq n-G_{max}+1$ is that $G_{max} \geq (n+1)/2$. Once more, G_{max} is a dictatorial group as in Case 2.3.

To summarize,

Theorem 8.28

Given a group formation $\mathbf{G}=(G_1, \ldots, G_g)$, $n=\Sigma G_i$, α, A, p;

Conditions 1G, 2G, 3, 4 may be compatible as in the following cases:

Case 1: $\alpha/A>(p-1)/p$ impossible case.

Case 2: $\alpha=(p-1)/p$

(a) when $\alpha g \leq A$ and $g \leq p$, a necessary and sufficient condition for compatibility is that

$G_{max} \geq (n+1)/2$

which is equivalent to G_{max} being a dictatorial group. i.e. it controls both a path and a cut.

(b) In all other cases a sufficient condition for compatibility is that $(p-1)n/p$ is not integer.

Case 3: $\alpha/A<(p-1)/p$

(a) When $\alpha g \leq A$ and $g \leq p$: as in Case 2(a) above.

(b) In all other cases the conditions may or may not be compatible, depending on the solution to Problems Q and T.

Proof: See discussion above. //

Example 8.29

When $G_i=1$ for $i=1,\ldots,g$ then we must be able to derive the conditions of compatibility for symmetric structures when there are no group formations:

For symmetric structures with no group formations we have shown that we may hope for compatibility only when $\alpha/A=(p-1)/p$, from which it follows that $\alpha=p-1$ and $A=p$.

Let again,

$$n=p\rho+\epsilon \quad \text{with } \rho\geq 0, \quad 0\leq\epsilon\leq p-1 \quad \text{and } \rho, \quad \epsilon$$

integers.

Note that $n=\Sigma G_i=g$, and assume $g\geq p$; thus $\rho\geq 1$.

When $G_i=1$ for all i, the optimal solution to Problem Q is $q^*=[[\alpha n/A]]$ and since $\alpha=p-1$, $A=p$,

$$q^*=[[(p-1)n/p]] \tag{8.31}$$

On the other hand, constraints (8.1), (8.2) of Problem T when $G_i=1$ for all i reduce to,

$$\mathbf{Te}\geq t\mathbf{e} \tag{8.32}$$

and

$$\mathbf{Te}-\mathbf{e}\leq(t-1)\mathbf{e} \quad \text{which simplifies to } \mathbf{Te}\leq t\mathbf{e} \tag{8.33}$$

It follows that

$$\mathbf{Te}=t\mathbf{e} \tag{8.34}$$

But then the optimal solution to Problem T is

$$t^*=n-[[n/p]]=[n-n/p]=[(p-1)n/p] \tag{8.35}$$

Also we require that

$$t^*+1\leq k\leq q^* \tag{8.36}$$

thus,

$$[(p-1)n/p]+1\leq k\leq [[(p-1)n/p]] \tag{8.37}$$

- If $(p-1)n/p$ is integer, (8.37) cannot hold and the conditions for logic and answers are incompatible.
- If $(p-1)n/p$ is not integer, (8.37) holds when,

$$k=[[(p-1)n/p]]=[(p-1)n/p]+1$$

and substituting for $n=p\rho+\epsilon$,

$$k=(p-1)\rho+[[(p-1)n/p]]$$

But $1\leq\epsilon\leq p-1$ thus, $\epsilon/p\leq 1$ and then,

$$k=(p-1)\rho+\epsilon$$

as expected by theory of Chapter 7.

Example 8.30

Let $A=p=5$, $\alpha=p-1=4$, $G_1=6$, $G_2=3$, $G_3=3$, $G_4=3$, $G_5=3$, $G_6=2$, $G_7=1$, then $n=21$. Solving Problem R and W we get $e_1=6$, $e_2=3$, $e_3=3$, $e_4=3+1$, $e_5=3+2$, $r^*=3$, $w^*=6$ and thus $q^*=n-r^*=18$, $u^*=n-w^*=15$. Now slight changes to $\mathbf{U}^*$ without altering u^* give us a solution to Problem T and thus $t^*=u^*=15$. Note also that $t^*=n-G_{max}$ and thus we are dealing with a one component cut structures.

Then from $t^*+1\leq k\leq q^*$ we obtain $16\leq k\leq 18$. Therefore any k/n structure with $n=21$ and $16\leq k\leq 18$ will satisfy the compatibility conditions. Further compare with the k/21 structures where no contraction takes place. By previous theory we know that the k/n structures must satisfy,

$$n=p\rho+\epsilon=4*5+1=21 \quad (p=4,\ \rho=5,\ \epsilon=1) \text{ and}$$

$$k=(p-1)\rho+\epsilon=3*5+1=16.$$

Thus, only the 16/21 structure out of the k/21 structures is compatible with Conditions 1G, 2G, 3, 4 when no group formation is

allowed.

Example 8.31

Let $p=5$, $A=p+1=6$, $\alpha=p-1=4$ and $G_1=6$, $G_2=3$, $G_3=3$, $G_4=3$, $G_5=3$, $G_6=2$, $G_7=1$. Solving Problem *R* we allocate the G_i's in $A=6$ sets $Z_1,\ldots,Z_6$ so that, $Z_1=6$, $Z_2=3$, $Z_3=3$, $Z_4=3$, $Z_5=3$, $Z_6=2+1=3$. Therefore, $r^*=3$ and $q^*=n-r^*=18$.

Solving Problem *W* we get, as in Example 8.30,

$e_1=6$, $e_2=3$, $e_3=3$, $e_4=3+1$, $e_5=3+2$, $w^*=6$, $u^*=n-w^*=15=t^*$ and finally the condition for compatibility $t^*+1\leq k\leq q^*$ becomes $16\leq k\leq 18$ for $n=21$.

Remark On p-Invariant Contractions

When a k/n structure is contracted according to **G** its p will change (p is non decreasing under contraction). There are cases where the contraction will preserve the same p. The straight forward method is to write down all min paths of k/n, apply contraction **G** and then check the min path matrix $\mathbf{P}_\varphi$ of the resulting structure φ, to see if any p or more rows have a common component. There is, however, another way using the theory just presented.

Let a group formation **G** which when embedded in the symmetric structure k/n ($n=\Sigma G_i$), results in a contracted structure φ so that $\varphi\in\mathcal{S}_p$ but $\varphi\notin\mathcal{S}_{p+1}$. Further it is required that $k/n\in\mathcal{S}_p$ but $k/n\notin\mathcal{S}_{p+1}$.

The requirement that,

$$t^*(p+1,\mathbf{G})\geq k\geq t^*(p,\mathbf{G})+1 \tag{8.38}$$

assures us that $\varphi\in\mathcal{S}_p$ and $\varphi\notin\mathcal{S}_{p+1}$

However, k must also satisfy,

$$((p-1)n+1)/p \leq k \leq (pn+1)/(p+1) \tag{8.39}$$

so that $k/n \in \mathcal{S}_p$ but $k/n \notin \mathcal{S}_{p+1}$. Therefore,

$$\max\{t^*(p, G), ((p-1)n+1)/p\} \leq k \leq \min\{t^*(p+1, G), (pn+1)/(p+1)\}$$

Since $t^*(p, G) \leq (p-1)n/p$ while $t^*(p+1, G) \leq pn/(p+1)$, we require,

$$((p-1)n+1)/p \leq k \leq pn/(p+1) \tag{8.40}$$

In order for (8.40) to be satisfied, there must be at least one integer between $((p-1)n+1)/p$ and $pn/(p+1)$. Taking the difference between those two quantities and requiring that it be greater or equal to 1, we obtain a nacessary condition for the satisfaction of (8.40),

$$n \geq (p+1)^2 \tag{8.41}$$

8.6 Duality: OR Logic Conditions

For reasons of symmetry we translate already exposed results on AND logic to OR logic by replacing as usual α by $\overline{\alpha}$, k by $n-k+1$, and talking about cuts instead of paths and $\overline{\mathcal{S}}_p$ instead of $\mathcal{S}_p$.

First Conditions 1G, 2G, 3, 4 take the following form:

Condition 1GD (non emptiness of rejection set)

-If $\overline{\alpha}g \geq A$ then $n-k+1 \leq \overline{q}^*$ or equivalently

$$k \geq n-\overline{q}^*+1=\overline{r}^*+1$$

where $\overline{q}^*$ is optimal for Problem $\overline{Q}$ and $\overline{r}^*$ is optimal for Problem $\overline{R}$.

-If $\overline{\alpha}g \leq A$ then $n-k+1 \leq G_{max}$ or equivalently,

$$k \geq n-G_{max}+1$$

Condition 2GD ($\varphi \in \overline{\mathcal{S}}_p$)

-If $g \geq p$ then $n-k+1 \geq t^*+1$ or

$$k \leq n-t^*$$

where t^* is optimal for Problem T.

-If $g \leq p$ then, $n-k+1 \geq n-G_{max}+1$ or $k \leq G_{max}$.

Condition 3D

$$\overline{\alpha} \leq p-1$$

Condition 4D

$$1 \leq \overline{\alpha} \leq A-1$$

where Problem $\overline{Q}$ is the same as Problem Q only $\overline{\alpha}$ replaces α. Problem $\overline{R}$ similarly. Problem U and W remain the same.

Finally the dual of Theorem 8.27 for the compatibility of Conditions 1GD, 2GD, 3D, 4D is:

Case 1: $\overline{\alpha}/A>(p-1)/p$ is impossible.

Case 2: $\overline{\alpha}/A=(p-1)/p$

(a) If $\overline{\alpha}g\leq A$ and $g\leq p$, a necessary and sufficient condition for compatibility is $G_{max}\geq(n+1)/2$ which is equivalent to G_{max} being a dictatorial group.

(b) In all other cases it is sufficient for compatibility to require that $(p-1)n/p$ is not integer.

Case 3: $\overline{\alpha}/A<(p-1)/p$

(a) As in Case 2(a).

(b) In all other cases Conditions 1GD, 2GD, 3D, 4D may or may not be compatible depending on the solution to Problems $\overline{Q}$ and T.

Chapter 9

CONDITIONS FOR THE EXISTENCE OF ANSWERS THAT OBEY LOGIC: A GENERALIZATION FOR COHERENT STRUCTURES

In Chapter 7 symmetric structures were studied and conditions were formulated and their compatibility examined to find those symmetric structures that for any given α, A, p they will guarantee the existence of answers that respect logic.

In Chapter 8 we searched for those symmetric structures that given α, A, p and a group formation (or party formation) **G**, will guarantee answers that respect logic. These families are wider than those of Chapter 7.

In the present chapter we will take any coherent structure that belongs to $\mathcal{S}_p$ (or $\overline{\mathcal{S}}_p$) and formulate conditions that will guarantee answers and logic consistency. Finally, we will connect to previous results in Chapters 7 and 8.

9.1 Formulation of Conditions

A necessary and sufficient condition for the existence of answers (non emptiness of $\mathcal{A}_\varphi$ or $\bar{\mathcal{A}}_\varphi$) can be formed for a coherent structure φ.

Consider the problem,

Problem *X*

Max α

so that

$$\mathbf{P}_\varphi \mathbf{Q}^T \le \Pi - [1] \tag{9.1}$$

$$\sum_{i=1}^{A} q_{ij} = \alpha \qquad \text{for } j = 1, \ldots, g \tag{9.2}$$

q_{ij} is 0,1 variable (It has the meaning that it is 1 when component j votes for issue i and 0 otherwise),

where

$$\Pi = \begin{bmatrix} \pi_1 \cdots\cdots \pi_1 \cdots\cdots \pi_1 \\ \vdots \qquad \vdots \qquad \vdots \\ \pi_m \cdots\cdots \pi_m \cdots\cdots \pi_m \end{bmatrix}$$

is a mxA matrix.

π_i is the number of components in the ith min path of φ (it is equal to the number of ones in the ith row of $\mathbf{P}_\varphi$).

Note that $\pi_i \ge 2$ otherwise there is a one component path and trivially there is always an answer.

$\mathbf{P}_\varphi$ is the min path matrix of φ (it is mxg)

$\mathbf{Q}$ is the matrix of variables q_{ij} (it is Axg)

[1] is a mxA matrix whose elements are all 1.

$\mathbf{Q}^T$ is the transpose of $\mathbf{Q}$

Problem X finds how much I can increase α (the number of alternatives each component is obliged to accept) and still obtain no answer under any voting realization $\mathbf{Q}$.
Note that $\mathbf{P}_\varphi\mathbf{Q}^T$ produces mA conditions checking each alternative (A in total) with each min path (m in total) and making sure that no alternative passes from any min path under the voting situation $\mathbf{Q}$. It follows that if α is chosen so that $\alpha \geq \alpha^*+1$, where α^* is the optimal solution to Problem X, which we will show it exists, there is no voting $\mathbf{Q}$ that will block all alternatives. We are therefore, guaranteed an answer when $\alpha \geq \alpha^*+1$. This shows the sufficiency of the condition. Necessity is also clear, because if to the contrary $\alpha \leq \alpha^*$, then there is a voting situation, namely $\mathbf{Q}^*$ (the optimal solution to Problem X) that results in $\mathcal{A}_\varphi = \emptyset$.

As far as respect of logic is needed, we know that if $\varphi \in \mathcal{S}_p$, a necessary and sufficient condition is that $\alpha \leq p-1$.

We conclude therefore, that the necessary and sufficient condition for existence of answers and respect of logic for $\varphi \in \mathcal{S}_p$ is that,

$$\alpha^*+1 \leq \alpha \leq p-1 \tag{9.3}$$

where α^* is the optimal solution to Problem X.

Observe that for OR logic consistency, when we demand the non emptiness of $\overline{\mathbb{A}}_\varphi$, we talk about $\mathbf{C}_\varphi$ instead of $\mathbf{P}_\varphi$ and for $\overline{\alpha}$ instead of α as usual.

Proposition 9.1

Problem X always has a solution.

Proof:

Pick the particular solution,

$\alpha'=0$ and $\mathbf{Q}'^T=[\mathbf{0}]$ (=gxA matrix of zeros)

then $\mathbf{P}_\varphi\mathbf{Q}'^T\leq\Pi-[\mathbf{1}]$ is satisfied since $\pi_i\geq 2$ for all i.

Also $\Sigma^A{}_{i=1}q'_{ij}=\alpha'=0$ is satisfied.

Thus a solution to Problem X always exists. //

Since the q_{ij}'s are 0,1 variables, Problem X is also bounded and hence an optimal solution α^*, $\mathbf{Q}^*$ always exists.

Problem X takes simpler forms when assumptions are made on φ, and therefore on $\mathbf{P}_\varphi$. In the following, such assumptions are imposed and the resulting form of Problem X found.

Proposition 9.2

If $\mathbf{P}_\varphi$ (and hence φ) is derivable from a symmetric structure by the sequence of operations K, OP (say from $K/\Sigma G_i$ by contraction K according to the group formation $\mathbf{G}$, and then omission of paths OP), then Problem X can take the following form:

Problem X_1

$$\text{Max } \alpha$$

$$[\mathbf{P}_\varphi \circ \mathbf{G}]\mathbf{Q}^T \leq (K-1)[\mathbf{1}] \tag{9.4}$$

$$\sum_{i=1}^{A} q_{ij} = \alpha \qquad \text{for } j=1,\ldots,g$$

where $[\mathbf{P}_\varphi \circ \mathbf{G}]$ is a mxg matrix whose elements, g_{ij}, are given by,

$$g_{ij} = G_j p_{ij} \tag{9.5}$$

where p_{ij}'s are the elements of $\mathbf{P}_\varphi$.

Proof:

Look first at $[\mathbf{P}_\varphi \circ \mathbf{G}]$

$$[\mathbf{P}_\varphi \circ \mathbf{G}] = \begin{bmatrix} G_1 p_{11} & \cdots\cdots & G_g p_{1g} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ G_1 p_{m1} & \cdots\cdots & G_g p_{mg} \end{bmatrix} \tag{9.6}$$

Take the first row of $[\mathbf{P}_\varphi \circ \mathbf{G}]$ and multiply it by the ith column of $\mathbf{Q}^T$ (inner product). We obtain,

$$\sum_{j=1}^{g} g_{j1} q_{ij}$$

which represents the number of votes that go to issue i by path 1 in the original mother structure $K/\Sigma G_i$.

If all components in path i vote for issue j it passes and $\Sigma^{g}_{j=1} g_{ji} q_{ij} = (\mathbf{P}_\varphi \mathbf{G})_i$ = the first element of $\mathbf{P}_\varphi \mathbf{G}$ and since $\mathbf{P}_\varphi$ is the result of (K, OP), we know that $\mathbf{P}_\varphi \mathbf{G} \geq k\mathbf{e}$ and thus $\Sigma g_{ji} q_{ij} \geq k$.

If, however, any one component in path i does not vote for issue j, then,

$\Sigma g_{ji} q_{ij} \leq (\mathbf{P}_\varphi{}' \mathbf{G})_i$ = the first element of the vector in parentheses. Where $\mathbf{P}_\varphi{}'$ is the "missing element matrix" of $\mathbf{P}_\varphi$. Since $\mathbf{P}_\varphi$ is the result of (K, OP), we know that $\mathbf{P}_\varphi{}' \mathbf{G} \leq (k-1)\mathbf{e}$.

It follows that, $\Sigma^{g}_{j=1} g_{ji} q_{ij} \leq k-1$.

From the above considerations it follows that,

$$[\mathbf{P}_\varphi \circ \mathbf{G}] \mathbf{Q}^T \leq (k-1)[\mathbf{1}] \tag{9.7}$$

is equivalent to condition (9.1) of Problem X when φ is the result of (K, OP) from the symmetric structure $k/\Sigma G_1$ by contraction according to $\mathbf{G}$. //

Problem X_1 is further simplified when we assume that φ is the result of pure contraction according to the group formation $\mathbf{G}$ from a $k/\Sigma G_1$ structure.

Proposition 9.3

If φ is the result of contraction only, say, $K(\mathbf{G} | (k/\Sigma G_1))$, then Problem X_1 takes the form,

Problem X_2

$$\text{Max } \alpha$$

$$QG \leq (k-1)e$$

$$\sum_{i=1}^{A} q_{ij} = \alpha \quad \text{for } j=1,\ldots,g$$

q_{ij} are 0,1 variables

where **Q** is a Axg matrix of q_{ij}'s.

Proof:

Structure φ behaves like a $k/\Sigma G_i$ structure whose components vote in groups $G_1,\ldots,G_g$. We need only to check that none of the A issues gets more than k-1 votes. This is guaranteed by setting,

$$QG \leq (k-1)e$$

where q_{ij} equals one when group j accepts issue i and zero otherwise. //

Problems X_1 and X_2 maximize α given k, where $k/\Sigma G_i$ is the symmetric mother structure of φ. For each particular k the maximal α is found. The reverse but equivalent problem is to find the maximal k for each particular α. Then Problem X_1 becomes,

Problem X_{11}

$$\text{Min } k$$

$$[\mathbf{P}_\varphi \circ \mathbf{G}]\mathbf{Q}^T \leq (k-1)[\mathbf{1}] \tag{9.9}$$

$$\sum_{i=1}^{A} q_{ij} = \alpha \qquad \text{for } j=1,\ldots,g$$

q_{ij} are 0, 1 variables

while Problem X_2 becomes,

Problem X_{22}

$$\text{Min } k$$

$$\mathbf{QG} \leq (k-1)\mathbf{e} \tag{9.10}$$

$$\sum_{i=1}^{A} q_{ij} = \alpha \qquad \text{for } j=1,\ldots,g$$

q_{ij} are 0, 1 variables

But then Problem X_{22} is the same as

Problem Q

$$\text{Min } q$$

$$\mathbf{QG} \leq q\mathbf{e}$$

$$\sum_{i=1}^{A} q_{ij} = \alpha \qquad \text{for } j=1,\ldots,g$$

q_{ij} are 0, 1 variables

which was studied in Chapter 8.

9.2 A Bound on α^*

Consider the min cuts of a coherent structure φ. We want to find maximal groups of cuts which have no common component between them (taken in two). Out of these groups pick the one(s) with the min number of cuts as its members. Let this number be y.

To be more exact,

Definition

Given a coherent structure φ, a set of min cuts $\{C_1,...,C_c\}$ is called *maximal non overlapping set of cuts* iff:

(a) $C_i \cap C_j = \emptyset$ for all $i,j \in \{1,...,c\}$

(b) For any cut C_i such that $i \notin \{1,...,c\}$ there exists $j \in \{1,...,c\}$ such that $C_i \cap C_j \neq \emptyset$.

Consider that (or those) maximal non overlapping set of cuts that has the minimum number of elements (cuts). This number is called y.

Proposition 9.4

A necessary condition for a coherent structure φ to have $\mathcal{A}_\varphi \neq \emptyset$ is that

$$y\bar{\alpha} \leq A-1 \tag{9.12}$$

Proof:

If to the contrary $y\bar{\alpha} \geq A$ then it is possible (by letting

each of the y cuts in the chosen maximal non overlapping set of cuts, block $\overline{\alpha}$ different statements) to block all A statements and thus make $\mathcal{A}_{\varphi}=\emptyset$. //

Note that (9.12) is equivalent to,

$$y<A/\overline{\alpha} <=> \alpha/A>(y-1)/y \qquad (9.13)$$

Proposition 9.5

Let α^* be the optimal solution to Problem *X* then,

$$\frac{y}{y-1}\leq\frac{\alpha^*}{A} \qquad (9.14)$$

Proof:

If α is chosen so that it equals α^*+1 then because of Problem *X* we know that $\mathcal{A}_{\varphi}\neq\emptyset$. However, because of (9.13) $y<A/\overline{\alpha}$ or $\alpha>A(y-1)/y$ and substituting α^*+1 for α, (9.14) is obtained. //

Condition $yA/\overline{\alpha}$ is necessary but not sufficient for $\mathcal{A}_{\varphi}$ to be non empty. This can be shown by an example that satisfies the condition but blocks all issues:

Example 9.6

Take a 2 out of 3 structure. Let A=5 and $\overline{\alpha}=4$. Then the min cuts of the structure are {1,2}, {1,3}, {2,3}. The max non overlapping sets contain at most one cut. Thus y=1. Indeed condition (9.13) is satisfied since

$$1=y<A/\overline{\alpha}=5/4$$

Take now the following voting situation:

component 1 blocks issues $\{a_1, a_2, a_3, a_4\}$

component 2 blocks issues $\{a_2, a_3, a_4, a_5\}$

component 3 blocks issues $\{a_3, a_4, a_5, a_1\}$

Then cut $\{1, 2\}$ blocks issues $\{a_2, a_3, a_4\}$

cut $\{1, 3\}$ blocks $\{a_1, a_3, a_4\}$

cut $\{2, 3\}$ blocks $\{a_3, a_4, a_5\}$

Therefore the structure blocks all issues.

Remark

Let us focus our attention to the k/n symmetric structures. It was shown in Chapter 7 that a necessary and sufficient condition for the existence of answers is that

$$k \leq \alpha\rho + \varepsilon - \sigma$$

or equivalently

$$k \leq [[n\alpha/A]]$$

or equivalently

$$n/(n-k+1) < A/\overline{\alpha} \qquad (9.16)$$

(recall (7.8), (7.15), (7.16)).

What is γ for a k/n structure?

$$\gamma = [n/(n-k+1)]$$

Therefore condition (9.13):

$$\gamma < A/\overline{\alpha}$$

which is necessary for the existence of answers for any coherent structure, is indeed very close to the necessary and sufficient condition for a k/n structure, which is expressed by (9.16).

BOOK 4

PROBABILITY ON COHERENT STRUCTURES

Chapter 10

PROBABILISTIC BEHAVIOUR OF COMPONENTS

In previous chapters component i was assumed to express his views (accept or reject an issue) through the binary function $x_i(a|A)$ which takes the value 1 when component i accepts statement a and 0 when he rejects it, given the set of alternatives A.

Suppose now that we don't know exactly which statement component i accepts, but we want to consider the probability that he chooses statement a_1 or a_2, or both etc when presented with a set of alternatives A.

In this chapter the probabilistic logical behaviour is studied at the component level to enable us to examine how it transfers at the structure level in the next chapter.

10.1 The Probabilities of Logical Consistency

The probabilities associated with component's i L-NAND, R-NAND, NAND, L-NOR, R-NOR, NOR consistencies are studied and their interrelations found.

For economy of notation define the following events,

Event c: $\Pi^{q}_{j=1}x_i(a_j)=1$, then Event $\overline{c}$: $\Pi^{q}_{j=1}x_i(a_j)=0$

Event d: $x_i(\text{NOT}(a_1\wedge\ldots\wedge a_q))=1$,

then Event $\overline{d}$: $x_i(\text{NOT}(a_1\wedge\ldots\wedge a_q))=0$

Event e: $\sqcup^{q}_{j=1}x_i(a_j)=1$, then Event $\overline{e}$: $\sqcup^{q}_{j=1}x_i(a_j)=0$

Event f: $x_i(\text{NOT}(a_1\vee\ldots\vee a_q))=1$,

then Event $\overline{f}$: $x_i(\text{NOT}(a_1\vee\ldots\vee a_q))=0$

The probability that component i is L-NAND consistent for the set of statements $(a_1,\ldots,a_q)$ is given by,

P[component i is L-NAND for $(a_1,\ldots,a_q)$]=

$$=P[c=>\overline{d}] \tag{10.1}$$

But $(c=>\overline{d})<=>(\overline{c}\vee\overline{d})$. Therefore,

$$=P[\overline{c}\vee\overline{d}] \tag{10.2}$$

Conditioning on c we obtain,

$$=P[\overline{c}\vee\overline{d}|c]P[c]+P[\overline{c}\vee\overline{d}|\overline{c}]P[\overline{c}] \tag{10.3}$$

$$=P[\overline{d}|c]P[c]+P[\overline{c}] \tag{10.4}$$

while conditioning on d we obtain

$$=P[\overline{c}|d]P[d]+P[\overline{d}] \tag{10.5}$$

Similarly,

P[component i is R-NAND for $(a_1,\ldots,a_q)$]=

$$=P[\overline{d}=>c]=P[d\vee c] \tag{10.6}$$

$$=P[d|\overline{c}]P[\overline{c}]+P[c] \tag{10.7}$$

$$=P[c|\overline{d}]P[\overline{d}]+P[d] \tag{10.8}$$

It then follows that,

$$P[i \text{ is R-NAND}]-P[i \text{ is L-NAND}]=P[c]-P[\overline{d}]=P[\overline{c}]-P[d] \tag{10.9}$$

Also,

$P[i$ is NAND consistent for $(a_1,...,a_q)]=$

$$=P[(d \vee c)\wedge(\overline{c}\vee\overline{d})]$$

$$=P[c|\overline{d}]P[\overline{d}]+P[\overline{c}|d]P[d]$$

or using the identity $P[A\cup B]=P[A]+P[B]-P[A\cap B]$,

$$=P[i \text{ is L-NAND}]+P[i \text{ is R-NAND}]-1 \tag{10.10}$$

$$=P[c\wedge\overline{d}]+P[\overline{c}\wedge d] \tag{10.11}$$

$$=P[(a_1,...,a_q)\in\hat{F}_{iq}]+P[(a_1,...,a_q)\in\hat{T}_{iq}] \tag{10.12}$$

The corresponding probabilities for the types of OR consistencies are presented below,

$P[i$ is R-NOR consistent for $(a_1,...,a_q)]=$

$$=P[\overline{e}=>f]=P[e\vee f] \tag{10.13}$$

$$=P[e|\overline{f}]P[\overline{f}]+P[f] \tag{10.14}$$

$$=P[f|\overline{e}]P[\overline{e}]+P[e] \tag{10.15}$$

$P[i$ is L-NOR consistent for $(a_1,...,a_q)]=$

$$=P[e=>\overline{f}]=P[\overline{e}\vee\overline{f}] \tag{10.16}$$

$$=P[\overline{e}|f]P[f]+P[\overline{f}] \tag{10.17}$$

$$=P[\overline{f}|e]P[e]+P[\overline{e}] \tag{10.18}$$

From (10.14), (10.15), (10.17), (10.18) we obtain,

$$P[i \text{ is R-NOR}]-P[i \text{ is L-NOR}]=P[e]-P[\overline{f}]=p[f]-P[\overline{e}] \tag{10.19}$$

$$P[i \text{ is NOR}]=P[(e\vee f)\wedge(\overline{e}\vee\overline{f})]=P[(\overline{e}=>f)\wedge(f=>\overline{e})]$$

$$=P[e\vee f]+P[\overline{e}\vee\overline{f}]-P[(e\vee f)\vee(\overline{e}\vee\overline{f})] \quad (10.20)$$

$$=P[i \text{ is R-NOR}]+P[i \text{ is L-NOR}]-1 \quad (10.21)$$

$$=P[e\wedge\overline{f}]+P[f\wedge\overline{e}] \quad (10.22)$$

$$=P[(a_1,\ldots,a_q)\in\check{T}_{iq}]+P[(a_1,\ldots,a_q)\in\check{F}_{iq}] \quad (10.23)$$

The conditional probability quantities $P[\overline{d}|c]$, $P[d|\overline{c}]$ etc. are essential in determining the probability that component i observes a type of logical consistency. Further, if for example $P[\overline{d}|c]=1$ then P[i is L-NAND]=1 and vice versa because of (10.4). And similarly for the other types of logical consistencies. But the quantities $P[\overline{d}|c]$, $P[d|\overline{c}]$ etc. are intuitively attractive for another reason. They present a behavioral characteristic of component i that may not depend on the particular set of statements $(a_1,\ldots,a_q)$. This possibility will be further explored later.

10.2 Unrestricted Passing Set

Component i is assumed not to be restricted in the number of statements he will accept or reject when presented with a set of alternatives A. The *joint probability* that component i accepts a subset of statements in A and rejects the rest is denoted by,

$$P[x_i(a_j|A)=z_{ij},\ j=1,\ldots,A] \qquad (10.24)$$

where

$$z_{ij}=\begin{cases} 1 & \text{if component i accepts statement j} \\ 0 & \text{otherwise} \end{cases}$$

The marginal probability, $P[x_i(a_1|A)=z_{i1}]$ is found by taking the summation of (10.24) over all possible values (0, 1) of $z_{i2}, z_{i3},\ldots,z_{iA}$.

If component i chooses alternative a_1 independently of the fact that he has already chosen (or rejected) statement a_2, then we say that statements a_1 and a_2 are *independent* for component i.

In this case,

$$P[x_i(a_1|A)=z_{i1}|x_i(a_2|A)=1]=$$
$$=P[x_i(a_1|A)=z_{i1}|x_i(a_2|A)=0]=$$
$$=P[x_i(a_1|A)=z_{i1}] \qquad (10.25)$$

Independence of statements in A for component i certainly holds when i is absolute. Recall that component i is *absolute* when $x_i(a|A)=x_i(a)$ for all a in A.

If component i considers statements $a_1,\ldots,a_q$ independent, his

joint probability distribution can be written in product form,

$$P[x_i(a_1|A)=z_{i1},\ldots,x_i(a_q|A)=z_{iq}]=$$
$$=\Pi^q{}_{j=1}P[x_i(a_j|A)=z_{ij}] \qquad (10.26)$$

In the following we will often use the shorter notation,

$$p_i(a_j|A)=P[x_i(a_j|A)=1]=P[a_j\in\mathcal{A}_i(A)] \qquad (10.27)$$

and

$$1-p_i(a_j|A)=P[x_i(a_j|A)=0]=P[a_j\in\bar{\mathcal{A}}_i(A)] \qquad (10.28)$$

Further, the probability that a group of q statements of A belongs to the outcome sets $\hat{\mathcal{A}}_{iq}$ or $\hat{\bar{\mathcal{A}}}_{iq}$ or $\hat{T}_{iq}$ or $\hat{C}_{iq}$ etc. take the form:

$$P[(a_1,\ldots,a_q)\in\hat{\mathcal{A}}_{iq}(A)]=P[\Pi^q{}_{j=1}x_i(a_j|A)=1] \qquad (10.29)$$

$$P[(a_1,\ldots,a_q)\in\hat{\bar{\mathcal{A}}}_{iq}(A)]=P[\Pi^q{}_{j=1}x_i(a_j|A)=0] \qquad (10.30)$$

where we assume that $C_{AND}(a_1\wedge\ldots\wedge a_q|A)=q$.

Also,

$$P[(a_1,\ldots,a_q)\in\hat{T}_{iq}]=$$
$$=P[\Pi^q{}_{j=1}x_i(a_j)=1 \text{ and } x_i(\mathrm{NOT}(a_1\wedge\ldots\wedge a_q))=0] \qquad (10.31)$$

where again we assume $C_{AND}(a_1\wedge\ldots\wedge a_q|A)=q$

The expressions for the other outcome sets are similar. Using now conditional probability, (10.31) can be expressed differently,

$$P[(a_1,\ldots,a_q)\in\hat{T}_{iq}]=$$
$$=P[(\mathrm{NOT}(a_1\wedge\ldots\wedge a_q))\in\mathcal{A}_i|a_j\in\mathcal{A}_i;j=1,\ldots,q]$$
$$P[a_j\in\mathcal{A}_i;j=1,\ldots,q] \qquad (10.32)$$

where it is assumed that $\mathrm{NOT}(a_1\wedge\ldots\wedge a_q)\in A$.

The first term on the RHS of (10.32) is directly related to the

probability that component i is L-NAND consistent. Namely, it equals to $P[\bar{d}|c]$, while the second term is $P[c]$, which were defined in section 10.1.

It is intuitively possible that $P[\bar{d}|c]$ for component i is a behavioral characteristic which is not related to the particular statements $(a_1,...,a_q)$ but only on their complexity q. We may then define $P[\bar{d}|c]$ to be a function of q only:

$$\hat{l}_i(q)=P[\bar{d}|c] \quad (10.33)$$

If indeed the assumption expressed in (10.33) holds, we expect $\hat{l}_i(q)$ to move towards 1/2 as q increases because it is humanly more probable to be contradictory by making random mistakes as complexity of statements increases.

Following similar lines of thought, the probabilities that component i assigns a group of statements $(a_1,...,a_q)$ to the rest of the outcome sets, can be expressed as,

$$P[(a_1,...,a_q)\in\hat{C}_{iq}]=P[d|c]P[\Pi^q{}_{j=1}x_i(a_j)=1] \quad (10.34)$$

$$=(1-\hat{l}(q))P[c] \quad (10.35)$$

$$P[(a_1,...,a_q)\in\hat{B}_{iq}]=P[\bar{d}|\bar{c}]P[\bar{c}] \quad (10.37)$$

$$=(1-\hat{r}_i(q))P[\bar{c}] \quad (10.38)$$

where we let

$$\hat{r}_i(q)=P[d|\bar{c}] \quad (10.40)$$

which is related to the R-NAND consistency of component i (recall (10.7)).

Similarly,

$$P[(a_1,...,a_q)\in\hat{F}_{iq}]=P[d|\bar{c}]P[\bar{c}] \quad (10.42)$$

$$=\hat{r}_i(q)P[\bar{c}] \quad (10.43)$$

When component i is R-NAND consistent, $\hat{r}_i(q)=1$ and the RHS of (10.38) reduces to zero as expected; and similarly for $\hat{l}_i(q)=1$ and L-NAND consistency.

For disjunction of statements component i assigns any group $(a_1,\ldots,a_q)$ to the outcome sets according to the following probabilities,

$$P[(a_1,\ldots,a_q)\in\check{T}_{iq}]=P[\overline{f}\mid e]P[e] \tag{10.45}$$

$$=\check{l}_i(q)P[e] \tag{10.46}$$

where we let

$$\check{l}_i(q)=P[\overline{f}\mid e]$$

which is related to the probability that component i is L-NOR (recall (10.18)).

Also,

$$P[(a_1,\ldots,a_q)\in\check{C}_{iq}]=P[f\mid e]P[e] \tag{10.48}$$

$$=(1-\check{l}_i(q))P[e] \tag{10.49}$$

$$P[(a_1,\ldots a_q)\in\check{B}_{iq}]=P[\overline{f}\mid\overline{e}]P[\overline{e}] \tag{10.51}$$

$$=(1-\check{r}_i(q))P[\overline{e}] \tag{10.52}$$

where $\check{r}_i(q)=P[f\mid\overline{e}]$ (10.53)

$$P[(a_1,\ldots,a_q)]=P[f\mid\overline{e}]P[\overline{e}] \tag{10.55}$$

$$=\check{r}_i(q)P[\overline{e}] \tag{10.56}$$

Again when $\check{r}_i(q)=1$, the RHS of (10.52) reduces to zero as expected because i is R-NOR consistent; and similarly when $\check{l}_i(q)=1$ then i is L-NOR consistent and the RHS of (10.49) becomes zero.

Remark 1

If we assume that $C_{AND}(a_1\wedge\ldots\wedge a_q|\mathcal{A}_i)=q$ it is reasonable to conclude that $a_1,\ldots,a_q$ are logically independent statements in the sense that none of them is implied by the conjunction or disjunction of the rest (otherwise $C_{AND}<q$); neither is anyone the negation of the conjunction or disjunction of any of the rest for the same reason. The same holds for $C_{OR}(a_1\vee\ldots\vee a_q|\bar{\mathcal{A}}_i)=q$.

Thus if $a_1,\ldots,a_q$ are independent statements for component i, we can write

$$
\begin{aligned}
P[c]&=P[\Pi^q{}_{j=1}x_i(a_j|A)=1]\\
&=\Pi^q{}_{j=1}P[x_i(a_j|A)=1]\\
&=\Pi^q{}_{j=1}p_i(a_j|A)
\end{aligned}
\qquad (10.57)
$$

and

$$
\begin{aligned}
P[\bar{e}]&=P[\sqcup^q{}_{j=1}x_i(a_j|A)=0]\\
&=P[\Pi^q{}_{j=1}(1-x_i(a_j|A))=1]\\
&=\Pi^q{}_{j=1}P[1-x_i(a_j|A)=1]\\
&=\Pi^q{}_{j=1}P[x_i(a_j|A)=0]\\
&=\Pi^q{}_{j=1}(1-p_i(a_j|A))
\end{aligned}
\qquad (10.58)
$$

Using (10.57) and (10.58) we have a way of calculating P[e] and P[c] whenever independence of statement is a valid assumption.

Remark 2

If q=1 then we recall that $\mathcal{A}_i=\hat{\mathcal{A}}_{iq}=\check{\mathcal{A}}_{iq}$, $\bar{\mathcal{A}}_i=\hat{\bar{\mathcal{A}}}_{iq}=\check{\bar{\mathcal{A}}}_{iq}$, $\hat{T}_{i1}=\check{T}_{i1}=T_i$, $\hat{C}_{i1}=\check{C}_{i1}=C_i$, $\hat{B}_{i1}=\check{B}_{i1}=B_i$, $\hat{F}_{i1}=\check{F}_{i1}=F_i$ and therefore,

$$
\begin{aligned}
P[a_j\in T_i]&=P[a_j\in\mathcal{A}_i \text{ and } \bar{a}_j\in\bar{\mathcal{A}}_i]\\
&=P[\bar{a}_j\in\bar{\mathcal{A}}_i|a_j\in\mathcal{A}_i]p_i(a_j|A)
\end{aligned}
\qquad (10.59)
$$

$$P[a_j \in C_1]=P[\overline{a}_j \in \mathcal{A}_1 \mid a_j \in \mathcal{A}_1]\,p_1(a_j \mid A) \tag{10.60}$$

$$P[a_j \in B_1]=P[\overline{a}_j \in \overline{\mathcal{A}}_1 \mid a_j \in \overline{\mathcal{A}}_1]\,(1-p_1(a_j \mid A)) \tag{10.61}$$

$$P[a_j \in F_1]=P[\overline{a}_j \in \mathcal{A}_1 \mid a_j \in \overline{\mathcal{A}}_1]\,(1-p_1(a_j \mid A)) \tag{10.62}$$

and the conditional probabilities are now related to the components probability of being L-NOT or R-NOT consistent.

Remark 3

Let $C_{\mathrm{AND}}((a_1 \wedge \ldots \wedge a_q) \mid \mathcal{A}_1)=q$ $(q \geq 2)$, then

(a) If $\mathrm{NOT}(a_1 \wedge \ldots \wedge a_q)$ also belongs to A then,

$$P[(a_1,\ldots,a_q) \in \hat{T}_{1q}]+P[(a_1,\ldots,a_q) \in \hat{C}_{1q}]+P[(a_1,\ldots,a_q) \in \hat{B}_{1q}]+ \\ +P[(a_1,\ldots,a_q) \in \hat{F}_{1q}]=1 \tag{10.63}$$

(b) If $\mathrm{NOT}(a_1 \wedge \ldots \wedge a_q) \notin A$ then (10.63) holds with ≤ 1 instead of equality. (recall (3.3), (3.4), (3.5), (3.6)).

10.3 Restricted Passing Set

In case the passing set $\mathcal{A}_i$ of component i is restricted, we assume that the number of statements that component i is obliged to choose from A is exactly α. As mentioned in section 7.2, if there is no group of α statements in A that component i can accept without violating his logic principles (for example L-NAND consistency or NAND consistency etc.), then the problem has no meaning. Assume, therefore, that restricting component i to choosing α statements does not force him to violate his logic principles.

First redefine,

$$x_i(a_1|A,\alpha)=\begin{cases}1 & \text{if component i accepts } a_1 \text{ when restricted to accept } \alpha \text{ statements from } A\\ 0 & \text{otherwise}\end{cases}$$

Certainly,

$$\sum_{j=1}^{A} x_i(a_j|A,\alpha)=\alpha \qquad (10.64)$$

Also

$$P\left[\prod_{K=1}^{\alpha} x_i(a_{j(K)}|A,\alpha)=1\right]=$$

$$=P[a_{j(K)}\in\mathcal{A}_i(A);K=1,\ldots,\alpha] \qquad (10.65)$$

denotes the probability that component i will pick the particular group of α statements as passing.

Remark

We may think of the set of alternatives A being transformed to a new set A' whose elements $\{a_j'\}$ are α-plets of alternatives from A. Then when $x_i(a_j'|A', \alpha')=1$, $\alpha'=1$, it means that component i accepts α-plet a_j' from set A' where he is restricted to accept one and only one element $(\alpha'=1)$.

Certainly,

$$x_i(a_1|A, \alpha) = \bigsqcup_{j\in J_1} x_i(a_j'|A, \alpha') \qquad (10.66)$$

where $J_k=\{j|a_j' \text{ contains } a_k\}$

then

$$P[x_i(a_1|A)=1] = \sum_{j\in J_1} P[x_i(a_j'|A', \alpha')=1] \qquad (10.67)$$

No cross terms exist in (10.67) since $x_i(a_j'|A, \alpha')$ equals to 1 for only one a_j'.

Observe that since component i will choose one and only one group of α statements, we can write

$$\Sigma\ \Pi^{\alpha}_{k=1} x_i(a_{j(k)}|A, \alpha) = 1 \qquad (10.68)$$

where the notation j(k) is used as usual instead of the double subscript notation j_k, and the summation is carried over all α-plets of indices $\{j_1, \ldots, j_\alpha\}$.

Taking now expectations of both sides of (10.68) we obtain,

$$\Sigma\ P[\Pi^{\alpha}_{k=1} x_i(a_{j(k)}|A, \alpha)=1]=1 \qquad (10.69)$$

which was expected as it is the summation of the joint probabilities defined in (10.65) which must add up to one. Note that the summation in (10.68) and (10.69) contains $\binom{A}{\alpha}$ terms because

it takes all groups of α indices from $\{1,\ldots,A\}$.

Using probability theory (marginal probability) we can obtain the expression for the probability that a group of q statements ($q\leq\alpha$) belongs to the passing set of component i when he accepts exactly α statements from A,

$$P[a_{j(k)}\in\mathcal{A}_i;k=1,\ldots,q]=\Sigma \quad P[a_{j(k)}\in\mathcal{A}_i;k=1,\ldots,\alpha],$$

where the summation is carried over all groups of α-q indices $\{j_{q+1},\ldots,j_\alpha\}$ picked from $\{1,\ldots,A\}-\{j_1,\ldots,j_q\}$.

The RHS of (10.70) contains $\binom{A-q}{\alpha-q}$ terms under the summation as it represents the summation over those α-plets that contain the q elements $a_{j(1)},\ldots,a_{j(q)}$.

Now summing both sides of (10.70) over all possible groups of q statements, we can prove the following proposition,

Proposition 10.1

Let $q\leq\alpha$

$$\Sigma\ P[a_{j(k)}\in\mathcal{A}_i;k=1,\ldots,q]=\binom{\alpha}{q} \qquad (10.71)$$

where the summation is carried over all possible groups of q statements in A.

Proof:

Summing (10.70) on both sides over all groups of q statements in A we have that:

The LHS contains $\binom{A}{q}$ terms under the summation.

The RHS contains $\binom{A}{q}\binom{A-q}{\alpha-q}$ terms under the double summation and each term refers to some α-plet of statements in A. But the terms in the RHS must be equally divided among groups of α statements. There are $\binom{A}{\alpha}$ such groups.

Therefore, each term referring to a particular group of α

statements must appear,

$$\frac{\binom{A}{q}\binom{A-q}{\alpha-q}}{\binom{A}{\alpha}} \quad \text{times}$$

which equals $\binom{\alpha}{q}$.

But now the RHS of (10.70) summed over all possible groups of q statements can be written as

$$\binom{\alpha}{q} \Sigma\ P[a_{j(k)} \in \mathcal{A}_1; k=1,\ldots,\alpha] \qquad (10.73)$$

where the summation is meant to be carried over all groups of α statements in A. But the summation in the above quantity is equal to 1, since it is the total probability that component 1 picks some group of α statements from A when he is obliged to do so. Thus the proposition is proved. //

In particular when q=1 (10.71) becomes

$$\Sigma^{A}_{j=1} P[a_j \in \mathcal{A}_1] = \alpha \qquad (10.74)$$

This was expected from (10.64).

Remark 1

The probability that a group of q statements belongs to one of the outcome sets must be reexamined in the present case when component 1 is restricted in accepting α statements exactly.

First observe that

$\hat{C}_{1q} = \hat{T}_{1q} = \emptyset$ for $q \geq \alpha+1$

and that

$\check{B}_{1q} = \check{F}_{1q} = \emptyset$ for $q \leq A-\alpha+1$

Let now $(a_1,\ldots,a_q)$ be a group of q statements with $C_{AND}(a_1 \wedge \ldots \wedge a_q | A) = q \leq \alpha-1$ then we can express,

$$P[(a_1,\ldots,a_q)\in\hat{T}_{iq}(A)]=$$
$$=P[x_i(\text{NOT}(a_1\wedge..\wedge a_q)\mid A,\alpha)=0\mid\Pi^q{}_{j=1}x_i(a_j\mid A,\alpha)=1]$$
$$P[\Pi^q{}_{j=1}x_i(a_j\mid A,\alpha)=1] \qquad (10.75)$$

The second term in the RHS of (10.75) can take the product form as in (10.57) and (10.58) when alternatives are independent. However, the first term on the RHS of (10.75) cannot be thought of being a characteristic of the component independent of the particular set A as in the case of unrestricted α, since the obligation to accept exactly α statements from the particular set A may force component i to be contradictory.

Remark 2

Is there some relation between the probability that component i chooses a group of statements between the restricted and unrestricted case? If the probabilities associated with the restricted case are simply the conditional probabilities of the unrestricted case when it is given that exactly α statements are accepted, then

$$P[\Pi^\alpha{}_{k=1}x_i(a_{j(k)}\mid A,\alpha)=1]=P[x_i(a_{j(k)}\mid A)=1;k=1,\ldots,\alpha \text{ and } x_i(a_l\mid A)=0 \text{ for } l\in A-\{j_1,\ldots,j_\alpha\} \mid \text{ exactly } \alpha \text{ statements are chosen from } A] \qquad (10.76)$$

where

$$P[\text{exactly } \alpha \text{ statements are chosen from } A]=$$
$$=\Sigma\ P[\Pi^\alpha{}_{k=1}x_i(a_{j(k)}\mid A)=1,\ x_i(a_l\mid a)=0 \text{ for } l\in A-\{j_1,\ldots,j_\alpha\}] \qquad (10.77)$$

where the summation is carried over all groups of α statements in A, $\{j_1,\ldots,j_\alpha\}$. If independence of statements is a valid assumption (10.77) takes the form,

P[exactly α statements are chosen]=

$$=\Sigma_s \Pi p_1(a|A)\Pi(1-p_1(a|A)) \qquad (10.78)$$

where s varies from 1 to $(^A{}_\alpha)$,

the first product form is carried over $a\in A_{s(\alpha)}$,

the second product form is carried over

$a\in A-A_{s(\alpha)}$, where the set $A_{s(\alpha)}$ is the sth

group of α statements, and there are $(^A{}_\alpha)$ such groups.

Now using the definition of conditional probability, P[X|Y]=P[X and Y]/P[Y] to evaluate (10.76), we know the denominator is given by (10.78). The numerator under the same assumption of independence is given by,

$$\Pi^\alpha{}_{k=1}p_1(a_{j(k)}|A)\Pi(1-p_1(a|A)) \qquad (10.80)$$

where the second product form is carried over

$a\in A-\{a_{j(1)},\ldots,a_{j(\alpha)}\}$.

Chapter 11

PROBABILISTIC BEHAVIOUR AT THE STRUCTURE LEVEL

11.1 Definitions and Discussion

The probabilistic behaviour at the component level has been studied in the previous chapter. The transition to the structure level is the natural step. In this chapter the probabilistic behaviour of the structure, based on the probabilistic behaviour of its components is studied . The line of thought resembles that of the last chapter.

The joint probability of structure φ passing a subset of statements in A and rejecting the rest is given by,

$$P[\varphi(\mathbf{x}(a_1|A))=z_1,\dots,\varphi(\mathbf{x}(a_A|A))=z_A] \qquad (11.1)$$

where

$$z_i=\begin{cases}1 & \text{if } \varphi \text{ accepts } a_i\\ 0 & \text{if } \varphi \text{ rejects } a_i\end{cases}$$

Therefore, $P[\varphi(\mathbf{x}(a_j|A))=z_i]$ may be found as the marginal probability derived from (11.1) by summing over all possible values (0, 1) of the z_i's.

If statements are independent for component i, then

$$P[x_i(a_1|A)=z_{i1}|x_i(a_2|A)=z_{i2}]=P[x_i(a_1|A)=z_{i1}] \quad (11.3)$$

for any statements a_1, a_2 in A. If this is true for all components then independence of statements carries over to the structure level and hence,

$$P[\varphi(\mathbf{x}(a_1|A))=z_1|\varphi(\mathbf{x}(a_2|A))=z_2]=$$
$$=P[\ \varphi(\mathbf{x}(a_1|A))=z_1] \qquad (11.4)$$

It follows that the joint probability expressed in (11.1) can take the simple form,

$P[\varphi(\mathbf{x}(a_j|A))=z_j;\ j=1,\ldots,A]=$

$$=\Pi^{A}{}_{j=1}P[\varphi(\mathbf{x}(a_j|A))=z_j] \qquad (11.5)$$

Thus the quantities that we are interested in are the marginal probabilities,

$P[\varphi(\mathbf{x}(a_j|A))=1]=$

$$=P[a_j\in\mathcal{A}_\varphi]$$

$$=E\varphi(\mathbf{x}(a_j|A)) \qquad (11.6)$$

The complementary probabilities are given by,

$P[\varphi(\mathbf{x}(a_j|A))=0]=$

$$=1-E\varphi(\mathbf{x}(a_j|A))$$

$$=P[a_j\in\overline{\mathcal{A}}_\varphi] \qquad (11.7)$$

If the components in φ act independently, then $E\varphi(\mathbf{x}(a_j|A))$ can be expressed as a function $h(.)$ of the probabilities $\{p_i(a_j|A)\}$ of the components, i.e. as a function of

$$\mathbf{p}(a_j|A)=(p_1(a_j|A),\ldots,p_n(a_j|A))$$

Then we can write,

$$E\varphi(\mathbf{x}(a_j|A))=h(\mathbf{p}(a_j|A)) \qquad (11.8)$$

From reliability theory, we know that $h(\mathbf{p})$, called the *reliability function*, is a multilinear function i.e. it is linear in each of its arguments p_i. Some of its properties are presented in section 12.1 below. Observe that $h(\mathbf{p}(a_j|A))$ is well defined even for the restricted case, when components are obliged to accept α statements exactly. In this case we will write

$$E\varphi(\mathbf{x}(a_j|A,\alpha))=h(\mathbf{p}(a_j|A,\alpha)) \qquad (11.9)$$

The probability that a group of q statements belongs to one of

the outcome sets can be defined as at the component level. For example,

$$P[(a_1,\ldots,a_q)\in\hat{T}_{\varphi q}]= \tag{11.10}$$

$$=P[a_j\in\mathcal{A}_\varphi;\ j=1,\ldots,q \text{ and } NOT(a_1\wedge\ldots\wedge a_q)\in\overline{\mathcal{A}}_\varphi]$$

where we assume that $(a_1\wedge\ldots\wedge a_q)$ has AND complexity equal to q and that $NOT(a_1\wedge\ldots\wedge a_q)$ is a statement in A.

Using conditional probability (11.10) can be expressed as

$$P[(a_1,\ldots,a_q)\in\hat{T}_{\varphi q}]=$$

$$=P[NOT(a_1\wedge\ldots\wedge a_q)\in\overline{\mathcal{A}}_\varphi\,|\,a_j\in\mathcal{A}_\varphi;\ j=1,\ldots,q]$$

$$P[a_j\in\mathcal{A}_\varphi;\ j=1,\ldots,q] \tag{11.11}$$

As usual the first term on the RHS of (11.11) indicates the probability with which φ respects L-NAND consistency for the particular group $(a_1,\ldots,a_q)$ in the set A. If the components are L-NAND consistent and $\varphi\in\mathcal{S}_p$, then φ will be L-NAND consistent as long as $q\leq p-1$ (recall Proposition 4.8). In this case, the first term on the RHS of (11.11) is equal to 1.

Remark 1

If the statements $a_1,\ldots,a_q$ are independent, then

$$P[a_j\in\mathcal{A}_\varphi;\ j=1,\ldots,q]=$$

$$=\Pi^q{}_{j=1}P[\varphi(\mathbf{x}(a_j|A))=1]=$$

$$=\Pi^q{}_{j=1}E\varphi(\mathbf{x}(a_j|A)) \tag{11.12}$$

and if the components are independent,

$$=\Pi^q{}_{j=1}h(\mathbf{p}(a_j|A)) \tag{11.13}$$

Observe that the requirement $C_{AND}(a_1\wedge\ldots\wedge a_q|\mathcal{A}_\varphi)=q$ implies that no statement in $(a_1,\ldots,a_q)$ implies or is implied by the conjunction or disjunction or their negation of the rest of them.

Then, it is reasonable to assume that there are cases where the

Similarly,

$$P[\sqcup^{q}_{j=1}\ \varphi(\mathbf{x}(a_j|A))=0]=$$

$$=P[\Pi^{q}_{j=1}(1-\varphi(\mathbf{x}(a_j|A))=1]$$

$$=\Pi^{q}_{j=1}P[\varphi(\mathbf{x}(a_j|A))=0] \qquad (11.14)$$

and if the components are independent,

$$=\Pi^{q}_{j=1}(1-h(\mathbf{p}(a_j|A))) \qquad (11.15)$$

Remark 2

As at the component level (recall (10.63), (10.64)),

$$P[\mathbf{b}\in\hat{T}_{\varphi q}]+P[\mathbf{b}\in\hat{C}_{\varphi q}]+P[\mathbf{b}\in\hat{B}_{\varphi q}]+P[\mathbf{b}\in\hat{F}_{\varphi q}]\leq 1 \qquad (11.16)$$

where $\mathbf{b}=(a_1,\ldots,a_q)$. If also $\text{NOT}(a_1\wedge\ldots\wedge a_q)\in A$ then (11.16) holds in equality form.

Remark 3

Because of their special form, structures in $\mathcal{S}_p$, $\bar{\mathcal{S}}_p$, $\boldsymbol{M}$ have some of their outcome sets empty and thus the related probabilities equal to zero. In particular, for $\varphi\in\boldsymbol{M}$ when the components are NAND consistent and $|\mathcal{A}_i|=1=|\bar{\mathcal{A}}_i|$, then $B_\varphi=C_\varphi=\emptyset$ and

$$P[a_i\in T_\varphi]+P[a_i\in F_\varphi]=1 \text{ for all } a_i\in A \qquad (11.17)$$

Which says that each a_i is either strongly passing or strongly rejected.

11.2 Symmetric Structures and NAND Consistent Components

The outcome sets $\hat{T}_{\varphi q}$, $\hat{C}_{\varphi q}$ etc. and the probability that a group of q statements $(a_1, \ldots, a_q)$ belongs to them was defined. However, to calculate them is quite a different task. In this section we examine these probabilities under the assumption that the components in the structure are NAND consistent and find a bound. Then under the further assumption that the structure is symmetric (k/n), exact formulas for these probabilities are determined both for the case when $|\mathcal{A}_i|$ is unrestricted and the case when $|\mathcal{A}_i| = \alpha$, for $i = 1, \ldots, n$.

In what follows we restrict our attention to $\hat{T}_{\varphi q}$ for reasons of economy. Similar results hold for all other outcome sets. Also for simplicity of notation we will use $\mathbf{x}(a_j)$ instead of $\mathbf{x}(a_j | A)$.

Proposition 11.1

If the components are NAND consistent then,

$$P[(a_1, \ldots, a_q) \in \hat{T}_{\varphi q}] \geq \tag{11.18}$$

$$\geq P[\varphi(\mathbf{x}(\text{NOT}(a_1 \wedge \ldots \wedge a_q)) = 0] P[\Pi^q{}_{j=1} \varphi(\mathbf{x}(a_j)) = 1]$$

Proof:

$$P[(a_1, \ldots, a_q) \in \hat{T}_{\varphi q}] =$$

$$= P[\Pi^q{}_{j=1} \varphi(\mathbf{x}(a_j)) = 1 \mid \varphi(\text{NOT}(a_1 \wedge \ldots \wedge a_q)) = 0]$$

$$P[\varphi(\mathbf{x}(\text{NOT}(a_1 \wedge \ldots \wedge a_q)) = 0] \tag{11.19}$$

It suffices to show that,

$$P[\Pi^q{}_{j=1} \varphi(\mathbf{x}(a_j)) = 1 \mid \varphi(\mathbf{x}(\text{NOT}(a_1 \wedge \ldots a_q)) = 0] \geq$$

$$\geq P[\Pi^q{}_{j=1} \varphi(\mathbf{x}(a_j)) = 1] \tag{11.20}$$

$\varphi(\mathbf{x}(\mathrm{NOT}(a_1\wedge\ldots\wedge a_q))=0$ implies that there is a cut that blocks $\mathrm{NOT}(a_1\wedge\ldots\wedge a_q)$. This means that each component i in the blocking cut has $x_i(\mathrm{NOT}(a_1\wedge\ldots\wedge a_q))=0$. But by R-NAND consistency of components

$$\Pi^q{}_{j=1}x_i(a_j)\geq 1-x_i(\mathrm{NOT}(a_1\wedge\ldots a_q))=1$$

Thus, $\Pi^q{}_{j=1}x_i(a_j)=1$ and therefore each component in the cut under investigation passes all of the statements $a_1,\ldots,a_q$. It is now clear, since φ is coherent, that the probability that the statements $a_1,\ldots,a_q$ pass φ is not smaller when we are given the fact that there is a group of components (the cut above) where each component passes all of the statements, than when this fact is not given. Thus (11.20) holds. //

Proposition 11.2

If the components are NAND consistent, independent and the statements $(a_1,\ldots,a_q)$ are independent then,

$$P[\varphi(\mathbf{x}(\mathrm{NOT}(a_1\wedge\ldots\wedge a_q))=0]=h^D(\Pi^q{}_{j=1}\mathbf{p}(a_j) \qquad (11.21)$$

and

$$P[\Pi^q{}_{j=1}\varphi(\mathbf{x}(a_j))=1]=\Pi^q{}_{j=1}h(\mathbf{p}(a_j)) \qquad (11.22)$$

where $h(\mathbf{p})$ is the reliability function of $\varphi(\mathbf{x})$ and $h^D(\mathbf{p})$ is the reliability function of the dual of $\varphi(\mathbf{x})$ i.e. of $\varphi^D(\mathbf{x})$; and where

$$\Pi^q{}_{j=1}\mathbf{p}(a_j)\equiv(\Pi^q{}_{j=1}p_1(a_j),\ldots,\Pi^q{}_{j=1}p_n(a_j))$$

Proof:

(a) $P[\Pi^q{}_{j=1}\varphi(\mathbf{x}(a_j))=1]=E[\Pi^q{}_{j=1}\varphi(\mathbf{x}(a_j))]$ (11.23)

because φ is binary. Now because of independence of a_j's,

$$=\Pi^q{}_{j=1}E\varphi(\mathbf{x}(a_j)) \qquad (11.24)$$

and because of independence of components,

$$=\Pi^q{}_{j=1}h(\mathbf{p}(a_j)) \qquad (11.25)$$

(b) $P[\varphi(\mathbf{x}(\mathrm{NOT}(a_1\wedge\ldots\wedge a_q))=0]=P[\varphi(1-\Pi^q{}_{j=1}\mathbf{x}(a_j))=0]$

because of NAND consistency of components

$$=P[\varphi^D(\Pi^q{}_{j=1}\mathbf{x}(a_j))=1] \tag{11,26}$$

Finally, independence of components implies,

$$=h^D(\Pi^q{}_{j=1}\mathbf{p}(a_j)) \tag{11.27}$$

and the proof is completed. //

Proposition 11.3

If the components are NAND consistent, independent and the statements are independent then,

$$P[(a_1,\ldots,a_q)\in\hat{T}_{\varphi q}]\geq h^D(\Pi^q{}_{j=1}\mathbf{p}(a_j))\Pi^q{}_{j=1}h(\mathbf{p}(a_j)) \tag{11.28}$$

Proof:

By Propositions 11.1, 11.2. //

11.2.1 Unrestricted Components' Passing Set

Let us now restrict our attention to symmetric structures (k/n) and study the form of $P[(a_1,\ldots,a_q)\in\hat{T}_{\varphi q}]$

(A) We start with the simplest case when q=1, then,

$$P[a_1\in T_\varphi]\equiv P[\varphi^D(\mathbf{x}(a_1))=1\,|\,\varphi(\mathbf{x}(a_1))=1]P[\varphi(\mathbf{x}(a_1))=1] \tag{11.29}$$

If $\varphi\in \mathcal{S}_2$ then the first term on the RHS of (11.29) is always equal to one. Since all symmetric structures belong to $\mathcal{S}_2\cup\bar{\mathcal{S}}_2$, it only remains to examine the case when $\varphi\in\bar{\mathcal{S}}_2$. But in this case $k\leq(n+1)/2$ (recall Example 1.1).

Proposition 11.4

Let $\varphi\in\bar{S}_2$ and the components be NAND consistent and independent then,

$$P[\varphi(\mathbf{x}(\bar{a}_1))=0|\varphi(\mathbf{x}(a_1))=1]=$$

$$=P[\varphi^D(\mathbf{x}(a_1))=1|\varphi(\mathbf{x}(a_1))=1]$$

$$=h^D(\mathbf{p}(a_1))/h(\mathbf{p}(a_1)) \tag{11.30}$$

Proof:

$$P[\varphi^D(\mathbf{x}(a_1))=1|\varphi(\mathbf{x}(a_1))=1]=$$

$$=\frac{P[\varphi(\mathbf{x}(a_1))=1 \text{ and } \varphi^D(\mathbf{x}(a_1))=1]}{P[\varphi(\mathbf{x}(a_1))=1]}$$

$$=\frac{P[\varphi(\mathbf{x}(a_1))=1|\varphi^D(\mathbf{x}(a_1))=1]P[\varphi^D(\mathbf{x}(a_1))=1]}{P[\varphi(\mathbf{x}(a_1))=1]} \tag{11.31}$$

but since $\varphi\in\bar{S}_2$,

$$P[\varphi(\mathbf{x}(a_1))=1|\varphi^D(\mathbf{x}(a_1))=1]=1 \tag{11.32}$$

This is true because $\varphi(\mathbf{x}(a_1))=\varphi(1-\mathbf{x}(\bar{a}_1))$ by NAND consistency of components. Then $\varphi(\mathbf{x}(a_1))=1$ <=> $1-\varphi(1-\mathbf{x}(\bar{a}_1))=0$ <=> $\varphi^D(\mathbf{x}(\bar{a}_1))=0$. But $\varphi\in\bar{S}_2$ <=> $\varphi^D\in S_2$ and we know that

$$P[\varphi^D(\mathbf{x}(a_1))=0|\varphi^D(\mathbf{x}(a_1))=1]=1$$

when $\varphi^D\in S_2$ and therefore (11.32) holds.

Now back to (11.31) using (11.32):

$$P[\varphi^D(\mathbf{x}(a_1))=1|\varphi(\mathbf{x}(a_1))=1]=$$

$$=\frac{P[\varphi^D(\mathbf{x}(a_1))=1]}{P[\varphi(\mathbf{x}(a_1))=1]} \tag{11.33}$$

and when the components are independent,

$$=h^D(\mathbf{p}(a_1))/h(\mathbf{p}(a_1))=(1-h(\mathbf{1}-\mathbf{p}(a_1)))/h(\mathbf{p}(a_1)) \tag{11.34}$$

as was required to show. //

Remark 1

If φ is k/n and if we let $p_i(a_j)=\pi_j$ for all i then we know that

$$h(\mathbf{p}(a_j))=\Sigma^n_{r=k}\binom{n}{r}\pi_j^r\ (1-\pi_j)^{n-r}$$

$$h^D(\mathbf{p}(a_j))=\Sigma^n_{r=n-k+1}\binom{n}{r}\pi_j^r(1-\pi_j)^{n-r}$$

Remark 2

The conditional probability

$$P[\varphi(\mathbf{x}(\overline{a}_1))=0|\varphi(\mathbf{x}(a_1))=1]$$

calculated above, is related to the probability that the structure φ is L-NOT consistent for a given statement a_1. If in particular, $\varphi\in\boldsymbol{M}$ then $\varphi^D=\varphi$ and then

$$P[\varphi^D(\mathbf{x}(a_1))=1|\varphi(\mathbf{x}(a_1))=1]=1$$

for any situation as expected because structures in $\boldsymbol{M}$ are NOT consistent.

(B) Take now the case q=2 (after that we will generalize to any q). In general if $\varphi\in\boldsymbol{S}_{q+1}$ then,

$$P[\varphi(\mathbf{x}(\mathrm{NOT}(a_1\wedge\ldots\wedge a_q)))=0|\Pi^q_{j=1}\varphi(\mathbf{x}(a_j))=1]=1$$

In this case it follows that,

$$P[(a_1,\ldots,a_q)\in\hat{T}_{\varphi q}]=P[\Pi^q_{j=1}\varphi(\mathbf{x}(a_j))=1]$$

In order to avoid this trivial result we will assume that structure φ is so that $\varphi\notin\boldsymbol{S}_{q+1}$. In particular, when φ is a k out of n symmetric structure, then the requirement $\varphi\notin\boldsymbol{S}_{q+1}$ means that

$$k(q+1)<nq+1 \qquad (11.35)$$

as dictated by Theorem 7.3.

Back to our problem: We want to calculate

$P[(a_1, a_2) \in \hat{T}_{\varphi 2}] =$

$= P[\varphi(\mathbf{x}(\mathrm{NOT}(a_1 \wedge a_2))) = 0 \text{ and } \varphi(\mathbf{x}(a_1))\varphi(\mathbf{x}(a_2)) = 1]$

$= \Sigma^n_{k(1),k(2)=1} P[\varphi(\mathbf{x}(\mathrm{NOT}(a_1 \wedge a_2))) = 0$ and

$\varphi(\mathbf{x}(a_1))\varphi(\mathbf{x}(a_2)) = 1 | k_1$ components vote for a_1, k_2 vote for $a_2]P[k_1$ vote for a_1, k_2 vote for $a_2]$

(11.36)

But when either of k_1 or k_2 are less than or equal to $k-1$, $\varphi(\mathbf{x}(a_1))\varphi(\mathbf{x}(a_2)) = 0$. Thus the summation above starts from $k_1, k_2 \geq k$. Also the statement $\varphi(\mathbf{x}(a_1))\varphi(\mathbf{x}(a_2)) = 1$ is always satisfied when $k_1, k_2 \geq k$ since φ is a k/n structure. Then (11.36) becomes,

$= \Sigma^n_{k(1),k(2)\geq k} P[\varphi(\mathbf{x}(\mathrm{NOT}(a_1 \wedge a_2))) = 0 | k_1 \text{ for } a_1, k_2 \text{ for } a_2]$

$P[k_1 \text{ for } a_1, k_2 \text{ for } a_2]$ (11.37)

The event $\varphi(\mathbf{x}(\mathrm{NOT}(a_1 \wedge a_2))) = 0$ is equivalent to $\varphi^D(\mathbf{x}(a_1)\mathbf{x}(a_2)) = 1$ because by NAND consistency of components

$$\mathbf{1} - \mathbf{x}(\mathrm{NOT}(a_1 \wedge a_2)) = \mathbf{x}(a_1)\mathbf{x}(a_2)$$

But $\varphi^D(\mathbf{x}(a_1)\mathbf{x}(a_2)) = 1$ means that there is a path of φ^D (or a cut of φ) that passes both a_1 and a_2. This means that there are $n-k+1$ or more components in φ that accept both a_1 and a_2. Therefore (11.37) takes the following form:

$P[(a_1, a_2) \in \hat{T}_{\varphi 2}] =$

$= \Sigma^n_{k(1),k(2)\geq k} P[$there are $n-k+1$ or more components accepting both a_1 and $a_2 | k_1$ vote for a_1, k_2 for $a_2]P[k_1$ for a_1, k_2 for $a_2]$

(11.38)

To help us visualize the situation, we can define the *voting matrix* $\mathbf{V} = \{v_{ij}\}$ where its elements v_{ij} are defined:

$$v_{ij} = \begin{cases} 1 & \text{if component } j \text{ accepts } a_i \\ \\ 0 & \text{if component } j \text{ rejects } a_i \end{cases}$$

V is then a Axn matrix whose elements are 0 or 1. Looking now at the conditional probability term on the RHS of (11.38), we are told that the first row of **V** has k_1 ones, the second row of **V** has k_2 ones and we want to calculate the probability that "ones" *match* in the two rows and in fact that there are n-k+1 or more such *matchings*.

Let us first calculate the probability that there are r such matchings in the two rows. ($r \leq A$)

Let M: the random variable indicating the number of matches of ones between the two rows.

$P[M=r|k_1$ ones in first row, k_2 ones in second row$]=$

$$=\frac{\binom{n-k(1)}{k(2)-r}\binom{k(1)}{r}}{\binom{n}{k(2)}} \qquad r \leq \min(k_1, k_2) \qquad (11.39)$$

The argument to justify (11.39) is as follows: When the k_1 ones of the first row are fixed we pick r of them (there are $\binom{k(1)}{r}$ possible ways). Once r ones of the second row are fixed to match those of the first row, then there are k_2-r ones to allocate in the positions that correspond to zeros in the first row (there are n-k_1 such positions). Thus the k_2-r ones can be allocated in

$$\binom{n-k(1)}{k(2)-r}$$

possible ways. Finally, the sample space is $\binom{n}{k(2)}$ which represents all possible ways to allocate k_2 ones in n positions. The

RHS of equation (11.39) is the well known hypergeometric distribution and can also be explained using the usual notion of an urn with n balls k_1 of which are red and the rest black. We draw k_2 balls and ask for the probability that out of the k_2 balls, r are red and the rest (k_2-r) are black.

Simple calculation shows that (11.39) is symmetric in k_1 and k_2 as is intuitively expected,

$$\frac{\binom{n-k(1)}{k(2)-r}\binom{k(1)}{r}}{\binom{n}{k(2)}}=$$

$$=\frac{k_1!k_2!(n-k_1)!(n-k_2)!}{n!r!(k_2-r)!(k_1-r)!(n-k_1-k_2+r)!} \tag{11.40}$$

For (11.39) to hold we also need that $n-k_1 \geq k_2-r$ or $r \geq k_1+k_2-n$ and that $r \leq k_1$, $r \leq k_2$, $r \geq 0$. All these can be combined into the single condition

$$\max(k_1+k_2-n,0) \leq r \leq \min(k_1,k_2) \tag{11.41}$$

From (11.39) we can calculate,

$P[M \geq l \mid k_1,k_2]=$

$$=\sum_{r=l}^{\min(k_1,k_2)} \frac{\binom{n-k(1)}{k(2)-r}\binom{k(1)}{r}}{\binom{n}{k(2)}} \tag{11.42}$$

which represents the $P[\varphi(\mathbf{x}(\mathrm{NOT}(a_1 \wedge a_2))=0 \mid k_1,k_2]$, where $l \geq k+k_2-n$ because of (11.42). Further, because of (11.41), if $l \leq \max(k_1+k_2-n,0)$, (11.42) will be equal to 1 since it represents the sum of the terms of the hypergeometric distribution.

To complete our study we have to substitute (11.42) with $l=n-K+1$ into (11.38)

$P[(a_1,a_2)\in\hat{T}_{\varphi 2}]=$

$$=\Sigma^n_{K(1),K(2)\geq K}\ P[M\geq \max(n-K+1,K_1+K_2-n)\mid K_1,K_2]\ P[K_1,K_2] \tag{11.43}$$

Observe that the LHS of (11.43) is $P[\overline{d}|c]$ in the short notation of section 10.1. While, $P[a_1,a_2\in A_\varphi]=P[c]=\Sigma_{K(1),K(2)\geq K}\ P[K_1,K_2]$.

We can then calculate $P[\overline{d}|c]$ or the probability that the structure is L-NAND consistent for a_1, a_2 by applying (10.4).

Finally, assuming that components are independent and statements are independent and letting

$$p_j=P[x_i(a_j)=1] \text{ for all i and j}$$

we can express

$P[K_1,K_2]=$

$$=\binom{n}{K(1)}p_1^{K(1)}(1-p_1)^{n-K(1)}\binom{n}{K(2)}p_2^{K(2)}(1-p_2)^{n-K(2)} \tag{11.44}$$

(C) To generalize to any q we move inductively using the rules of conditional probability. Let

M_i: the number of matches among i rows.

For $q=3$

$P[M_3=r_3|K_1,K_2,K_3]=$

$$=\sum_{r_2=r_3}^{\min(k_1,K_2)} P[M_3=r_3|M_2=r_2,K_1,K_2,K_3]P[M_2=r_2|K_1,K_2,K_3] \tag{11.45}$$

$$\min(k_1,k_2)$$
$$= \sum_{r_2=r_3} \frac{\binom{n-r(2)}{k(3)-r(3)}\binom{r(2)}{r(3)}\binom{n-k(1)}{k(2)-r(2)}\binom{k(1)}{r(2)}}{\binom{n}{k(3)}\binom{n}{k(2)}} \quad (11.46)$$

where,

$$\max(r_2+k_3-n, 0) \le r_3 \le \min(r_2, k_3)$$

$$\max(k_1+k_2-n, 0) \le r_2 \le \min(k_1, k_2)$$

For any q,

$$P[M_q=r_q|k_1,\ldots,k_q]=$$

$$= \sum_{r_{q-1}=r_q}^{\min(k_1,\ldots,k_{q-1})} \frac{\binom{n-r(q-1)}{k(q)-r(q)}\binom{r(q)-1}{r(q)}}{\binom{n}{k(q)}}$$

$$\sum_{r_{q-2}=r_{q-1}}^{\min(k_1,\ldots,k_{q-2})} \frac{\binom{n-r(q-2)}{k(q-1)-r(q-1)}\binom{r(q-2)}{r(q-1)}}{\binom{n}{k(q-1)}}\ldots$$

....

$$\ldots \sum_{r_3=r_4}^{\min(k_1,k_2,k_3)} \frac{\binom{n-r(3)}{k(4)-r(4)}\binom{r(3)}{r(4)}}{\binom{n}{k(4)}}$$

$$\sum_{r_2=r_3}^{\min(k_1,k_2)} \frac{\binom{n-r(2)}{k(3)-r(3)}\binom{r(2)}{r(3)}\binom{n-k(1)}{k(2)-r(2)}\binom{k(1)}{r(2)}}{\binom{n}{k(3)}\binom{n}{k(2)}}$$

(11.47)

where

$$\max(r_{s-1}+k_s-n, 0) \le r_s \le \min(k_1,\ldots,k_s) \text{ for } s=2,3,\ldots,q.$$

From (11.47) we can calculate $P[M_q \geq n-k+1 | k_1, \ldots, k_q]$ and then use it to find,

$P[(a_1, \ldots, a_q) \in \hat{T}_{\varphi q}] =$

$$=\Sigma^{n}{}_{k(1),\ldots,k(q) \geq k} P[M_q \geq n-k+1 | k_1, \ldots, k_q] P[k_1, \ldots, k_q] \qquad (11.48)$$

If the components are independent and the statements are independent,

$$P[k_1, \ldots, k_q] = \Pi^{q}{}_{j=1} \binom{n}{k(j)} p_j^{k(j)} (1-p_j)^{n-k(j)} \qquad (11.49)$$

where $p_j = P[x_i(a_j)=1]$ for all i, j

Finally, the probability of φ being L-NAND consistent can be found by first calculating $P[\bar{d}|c]$ and then using (10.4). Observe that $P[\bar{d}$ and $c]$ is given by (11.48) while $P[c]$ is given by (11.49) and therefore $P[\varphi$ is L-NAND$] = P[\bar{d}$ and $c] + P[\bar{c}]$.

Remark

If we take the expected value of both sides in (11.48) where we use (11.49) for $P[k_1, \ldots, k_q]$, then an average value of $P[\bar{d}$ and $c]$, $E\{P[\bar{d}$ and $c]\}$, can be determined over the p_j's as they vary from 1 to 0. From this quantity the average probability of the structure φ being L-NAND can be calculated giving us a feeling of the general behaviour of the structure. If in particular we assume the p_j's are uniformly distributed over $[0,1]$ then from (11.48)

$E\{P[\bar{d}$ and $c]\} =$

$$=\Sigma\, P[M_q \geq n-k+1 | k_1, \ldots, k_q] \int_{(0,1)} P[k_1, \ldots, k_q] dp_1 \ldots dp_q$$

and using (11.49) and the definition of beta functions, we obtain,

$$=1/(n+1)^{q}\ \Sigma\, P[M_q \geq n-k+1 | k_1, \ldots, k_q]$$

where the summation is carried over $k_1, \ldots, k_q \geq k$ up to n, and hence,

$E\{P[\varphi \text{ is L-NAND}]\}=E\{P[\overline{d} \text{ and } c]\}+1-E\{P[c]\}=$

$=E\{P[\overline{d} \text{ and } c]+1-1/(n+1)^{q}.$

Example 11.1

Suppose we are given φ which is symmetric k/n with $\varphi\in\overline{S}_p$ $(k\leq(n+1)/2)$. Let in particular n=15 and let k take the values k=7, 5, 3, 1. Suppose that we are given a statement a_1 and we care to examine the probability that the structure φ is L-NOT consistent as the probability, p, of each component passing the statement a_1 varies between 0 and 1.

By Proposition 11.4 we know that $P[\overline{d}|c]$ is equal to

$P[\varphi(\mathbf{x}(\overline{a}_1))=0|\varphi(\mathbf{x}(a_1))]=h^D/h$

Using the binomial distribution to calculate $h^D(p)$ and $h(p)$, we obtain,

p-values	7/15	5/15	3/15	1/15
0.000	0.0000000	0.0000000	0.0000000	0.0000000
0.200	0.0431894	0.0000609	0.0000000	0.0000000
0.300	0.1162117	0.0013828	0.0000115	0.0000000
0.500	0.4359976	0.0629591	0.0036936	0.0000300
0.700	0.8823063	0.5158356	0.1268313	0.0047500
0.800	0.9827065	0.8357784	0.3980200	0.0351800
0.950	1.0000000	0.9999000	0.9638000	0.4632900
1.000	1.0000000	1.0000000	1.0000000	1.0000000

Table 11.1: The values of $P[\overline{d}|c]$ for the structures 7/15, 5/15, 3/15, 1/15 and for a single statement (q=1)

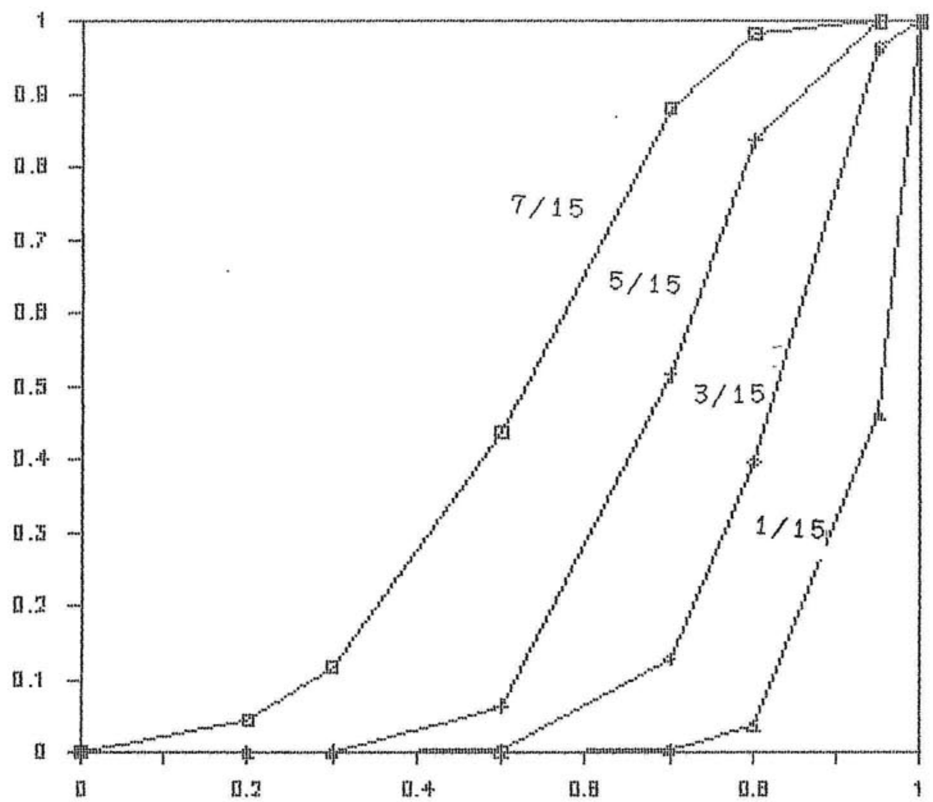


Figure 11.2: $P[\bar{d}|c]$ for the structures 7/15, 5/15, 3/15, 1/15 for a single statement (q=1)

From Fig. 11.2 we can see that as k increases the structure behaves more consistently because $P[\bar{d}|c]$ increases. Also for $k \geq 8$ the structure belongs to $\mathcal{S}_2$ and hence $P[\bar{d}|c]=1$ regardless of p.

Now from the conditional probability $P[\bar{d}|c]$ we can calculate the probability that the structure is L-NOT consistent. From (10.4),

$$P[\varphi \text{ is L-NOT}]=P[\bar{d}|c]P[c]+P[\bar{c}]$$

But

$P[c]=P[\varphi(\mathbf{x}(a_1))=1]=h(p)$

Thus

$P[L\text{-}NOT]=h^D-h+1$

Also the probability that we get both answers and logic consistency is given by $P[\bar{d}$ and $c]=P[\bar{d}|c]P[c]$. These probabilities are calculated in the following Table 11.3 and presented in the Figures 11.4a, 11.4b below.

Observe that P[L-NOT] is symmetric and it has a minimum at p=0.5. For $k\geq 8$ it is equal to 1 for all p because the structures belong to $\mathcal{S}_2$.

Figure 11.3 indicates that the system with the highest probability of both answers and logic consistency for single statements (q=1) is the majority structure (8/15) and decreases symmetrically as we move away on either side of 8/15. This intuitively nice property however does not hold for more than one statement as we will see in Example 11.2 and figures 11.17 and 11.18 below.

p-values	P[c] for 9/15 h9)15	P[c] for 7/15 h7)15	$P[\bar{d}|c]$ for 7/15 h9/h7	P[L-NOT] for 7/15 h9-h7+1
0.000	0.0000000	0.0000000	0.0000000	1.0000000
0.200	0.0007800	0.0180600	0.0431894	0.9827200
0.300	0.0152400	0.1311400	0.1162117	0.8841000
0.500	0.3036200	0.6963800	0.4359976	0.6072400
0.700	0.8688600	0.9847600	0.8823063	0.8841000
0.800	0.9819400	0.9992200	0.9827065	0.9827200
0.950	1.0000000	1.0000000	1.0000000	1.0000000
1.000	1.0000000	1.0000000	1.0000000	1.0000000

P[c] for 11/15 h11)15	P[c] for 5/15 h5)15	$P[\bar{d}|c]$ for 5/15 h11/h5	P[L-NOT] for 5/15 h11-h5+1
0.0000000	0.0000000	0.0000000	1.0000000
0.0000100	0.1642300	0.0000609	0.8357800
0.0006700	0.4845100	0.0013828	0.5161600
0.0592300	0.9407700	0.0629591	0.1184600
0.5154900	0.9993300	0.5158356	0.5161600
0.8357700	0.9999000	0.8358536	0.8358700
0.9999000	1.0000000	0.9999000	0.9999000
1.0000000	1.0000000	1.0000000	1.0000000

P[c] for 13/15	P[c] for 3/15	$P[\bar{d}\|c]$ for 3/15	P[L-NOT] for 3/15
h13)15	h3)15	h13/h3	h13-h3+1
0.0000000	0.0000000	0.0000000	1.0000000
0.0000000	0.6019800	0.0000000	0.3980200
0.0000100	0.8731700	0.0000115	0.1268400
0.0036900	0.9963100	0.0037037	0.0073800
0.1268300	0.9999000	0.1268427	0.1269300
0.3980200	1.0000000	0.3980200	0.3980200
0.9638000	1.0000000	0.9638000	0.9638000
1.0000000	1.0000000	1.0000000	1.0000000

P[c] for 15/15	P[c] for 1/15	$P[\bar{d}\|c]$ for 1/15	P[L-NOT] for 1/15
h15)15	h1)15	h15/h1	h15-h1+1
0.0000000	0.0000000	0.0000000	1.0000000
0.0000000	0.9648200	0.0000000	0.0351800
0.0000000	0.9952500	0.0000000	0.0047500
0.0000300	0.9999700	0.0000300	0.0000600
0.0047500	1.0000000	0.0047500	0.0047500
0.0351800	1.0000000	0.0351800	0.0351800
0.4632900	1.0000000	0.4632900	0.4632900
1.0000000	1.0000000	1.0000000	1.0000000

p-values	P[c] for 8/15 h8)15
0.000	0.0000000
0.200	0.0042400
0.300	0.0500100
0.500	0.5000000
0.700	0.9499900
0.800	0.9957000
0.950	1.0000000
1.000	1.0000000

Table 11.3: P[c], $P[\bar{d}|c]$, P[L-NOT] calculated for the structures 1/15, 3/15, 5/15, 8/15, 9/15, 11/15, 13/15, 15/15 .

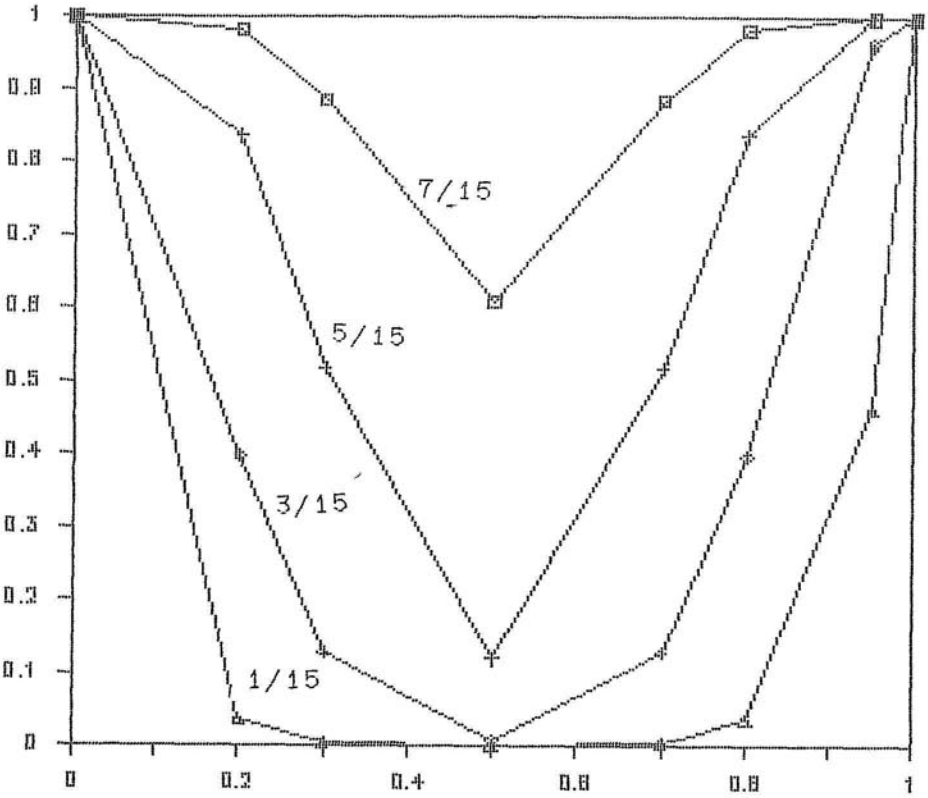


Figure 11.4a: P[L-NOT] for the structures 1/15, 3/15, 5/15, 7/15

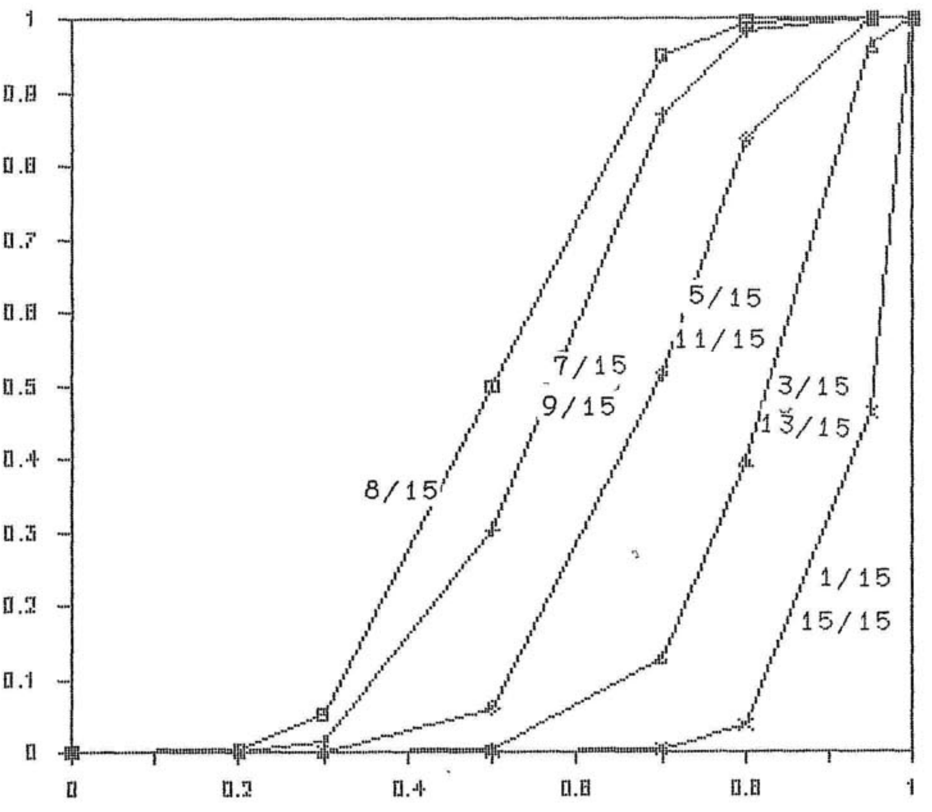


Figure 11.4b: P[$\overline{d}$ and c] for the structures 1/15, 3/15, 5/15, 7/15, 8/15, 9/15, 11/15, 13/15, 15/15.

Example 11.2

Let us study the probability that a k/n structure is L-NAND consistent for q=2. According to the theory developed,

P[$\overline{d}$ and c]=

$$= \sum_{k_1, k_2 \geq k}^{n} P[M \geq n-k+1 | k_1, k_2] P[k_1, k_2]$$

where $P[M \geq n-k+1 | k_1, k_2]$ is given by (11.42) with r ranging from $\max\{n-k+1, k_1+k_2-n\}$ to $\min\{k_1, k_2\}$,

$P[c]=P[k_1 \geq k, k_2 \geq k]=\Sigma^n_{k(1), k(2) \geq k} P[k_1, k_2]$

where $P[k_1, k_2]$ is given by (11.49)

P[L-NAND]=P[$\overline{d}$ and c]+P[$\overline{c}$]=P[$\overline{d}$ and c]-P[c]+1. And setting $p_1=p_2=p$ we plot: (a) P[c], the probability that statements a_1 and a_2 pass the structure, (b) P[$\overline{d}$|c], the probability that the negation of the conjunction of statements a_1, a_2 does not pass the structure given that a_1 and a_2 pass individually, (c) P[L-NAND], the probability that the structure respects L-NAND consistency for statements a_1, a_2, for the symmetric structures 1/15, 3/15, 5/15, 7/15, 8/15, 9/15, 10/15, 11/15, 13/15, 15/15 and 14/30 as p varies from 0 to 1. The results of the calculations appear in Table 11.5 and Figures 11.6 to 11.18.

We observe that as P[c] increases with p, P[L-NAND] starts from 1 and first decreases (since silence means consistency, "silence is golden") and after reaching a minimum it increases back to 1 as the probability of accepting both a_1, a_2 approaches 1. Also P[$\overline{d}$|c] starts from 0 and increases to 1 having an s-shape and becoming identical to the curve for P[L-NAND] as we approach p=1. For structures 11/15, up to 15/15 P[$\overline{d}$|c]=P[L-NAND]=1 for all p

because they belong to S_3 and we have two statements a_1, a_2.

Comparing now structures 1/15 to 15/15 in Figures 11.6 to 11.15, we see that 1/15 is a "talkative" structure (it has higher probability of giving answers) as the fast rise of P[c] indicates. But P[L-NAND] remains small for a wide range of p values indicating the low consistency of the system. Thus high probability of answers is accompanied with high probability of inconsistency. As we move to 3/15, 5/15 etc., the system is less "talkative" and also more consistent.

A comparison of 7/15 with 14/30 in Figure 11.16, shows that the curves become steeper as the population, n, increases but the ratio k/n remains the same.

As we already noted in the Remark preceding Example 11.1, a measure of the structure's performance both for giving answers and being consistent is given by $E\{P[\overline{d} \text{ and } c]\}$ where the expectation is carried over p which is assumed to be uniformly distributed over (0, 1). It was found that,

$E\{P[\overline{d} \text{ and } c]\} =$

$$= \frac{1}{(n+1)^q} \int P[\overline{d} \text{ and } c]\, dp$$

where the integral is over (0, 1) and where q=2 in the present example.

Comparing the curves $P[\overline{d}$ and c] for the different structures in Figures 11.17 and 11.18, we see that 9/15 is the one with the largest area underneath it while 10/15 and 8/15 (the odd majority

structure) are the second and third best respectively. This tells us that 9/15 structure gives a higher probability of having answers with logic for two statements and their logic conjunction.

Table 11.5

p-values	1/15 P[c]	1/15 P[NOTd|c]	1/15 P[L-NAND]	1/15 P[NOTd and c]
0.00000	0.00000	0.00000	1.00000	0.00000
0.05000	0.28800	0.00000	0.71190	0.00000
0.10000	0.63060	0.00000	0.36930	0.00000
0.30000	0.99050	0.00000	0.00947	0.00000
0.50000	0.99990	0.00000	0.00000	0.00000
0.70000	0.99999	0.00002	0.00002	0.00002
0.80000	0.99999	0.00124	0.00124	0.00124
0.90000	1.00000	0.04239	0.04239	0.04239
0.95000	1.00000	0.21463	0.21463	0.21463
0.99000	1.00000	0.73970	0.73970	0.73970
1.00000	1.00000	1.00000	1.00000	1.00000

3/15 P[c]	3/15 P[NOTd|c]	3/15 P[L-NAND]	3/15 P[NOTd and c]
0.00000	0.00000	1.00000	0.00000
0.00000	0.00000	0.99990	0.00000
0.03387	0.00000	0.96610	0.00000
0.76240	0.00000	0.23750	0.00000
0.99260	0.00000	0.00737	0.00000
0.99990	0.00294	0.00295	0.00294
0.99990	0.05281	0.05281	0.05280
1.00000	0.43640	0.43640	0.43640
0.99990	0.82540	0.82540	0.82532
1.00000	0.99700	0.99700	0.99700
1.00000	1.00000	1.00000	1.00000

5/15 P[c]	5/15 P[NOTd|c]	5/15 P[L-NAND]	5/15 P[NOTd and c]
0.00000	0.00000	1.00000	0.00000
0.00000	0.00000	1.00000	0.00000
0.00016	0.00000	0.99980	0.00000
0.23470	0.00000	0.76520	0.00000
0.88500	0.00013	0.11500	0.00012
0.99860	0.05067	0.05190	0.05060
0.99990	0.32220	0.32220	0.32217
0.99990	0.86057	0.86057	0.86048
1.00000	0.99990	0.99990	0.99990
1.00000	1.00000	1.00000	1.00000
1.00000	1.00000	1.00000	1.00000

7/15	7/15	7/15	7/15
P[c]	P[NOTd¦c]	P[L-NAND]	P[NOTd and c]
0.00000	0.00000	1.00000	0.00000
0.00000	0.00000	1.00000	0.00000
0.00000	0.00000	0.99999	0.00000
0.01798	0.00000	0.98280	0.00000
0.48490	0.00864	0.51920	0.00419
0.96970	0.28530	0.30690	0.27666
0.96974	0.72890	0.72940	0.70685
0.99990	0.98630	0.98630	0.98620
1.00000	0.99990	0.99990	0.99990
1.00000	1.00000	1.00000	1.00000
1.00000	1.00000	1.00000	1.00000

8/15	8/15	8/15	8/15
P[c]	P[NOTd¦c]	P[L-NAND]	P[NOTd and c]
0.00000	0.00000	1.00000	0.00000
0.00000	0.00000	1.00000	0.00000
0.00000	0.00053	0.99990	0.00000
0.00250	0.00619	0.99750	0.00002
0.25000	0.06919	0.76720	0.01730
0.90240	0.51920	0.56610	0.46853
0.99150	0.87720	0.87820	0.86974
1.00000	0.99000	0.99000	0.99000
1.00000	1.00000	1.00000	1.00000
1.00000	1.00000	1.00000	1.00000
1.00000	1.00000	1.00000	1.00000

9/15	9/15	9/15	9/15
P[c]	P[NOTd¦c]	P[L-NAND]	P[NOTd and c]
0.00000	0.00000	1.00000	0.00000
0.00000	0.00000	1.00000	0.00000
0.00000	0.14300	1.00000	0.00000
0.00023	0.23130	0.99980	0.00005
0.09218	0.43590	0.94800	0.04018
0.75490	0.82290	0.75490	0.62121
0.96420	0.96890	0.96420	0.93421
0.99930	0.99960	0.99930	0.99890
1.00000	1.00000	1.00000	1.00000
1.00000	1.00000	1.00000	1.00000
1.00000	1.00000	1.00000	1.00000

10/15 P[c]	10/15 P[NOTd!c]	10/15 P[L-NAND]	10/15 P[NOTd and c]
0.00000	0.00000	1.00000	0.00000
0.00000	0.60000	1.00000	0.00000
0.00000	0.92420	1.00000	0.00000
0.00001	0.94410	0.99990	0.00001
0.02276	0.96900	0.99920	0.02205
0.52070	0.99310	0.99640	0.51711
0.88160	0.99890	0.99910	0.88063
0.99550	0.99990	0.99990	0.99540
1.00000	1.00000	1.00000	1.00000
1.00000	1.00000	1.00000	1.00000
1.00000	1.00000	1.00000	1.00000

11/15 P[c]	11/15 P[NOTd!c]	11/15 P[L-NAND]	11/15 P[NOTd and c]
0.00000	1.00000	1.00000	0.00000
0.00000	1.00000	1.00000	0.00000
0.00000	1.00000	1.00000	0.00000
0.00000	1.00000	1.00000	0.00000
0.00350	1.00000	1.00000	0.00350
0.26570	1.00000	1.00000	0.26570
0.69850	1.00000	1.00000	0.69850
0.97470	1.00000	1.00000	0.97470
0.99870	1.00000	1.00000	0.99870
1.00000	1.00000	1.00000	1.00000
1.00000	1.00000	1.00000	1.00000

13/15 P[c]	13/15 P[NOTd!c]	13/15 P[L-NAND]	13/15 P[NOTd and c]
0.00000	1.00000	1.00000	0.00000
0.00000	1.00000	1.00000	0.00000
0.00000	1.00000	1.00000	0.00000
0.00000	1.00000	1.00000	0.00000
0.00001	1.00000	1.00000	0.00001
0.01608	1.00000	1.00000	0.01608
0.15840	1.00000	1.00000	0.15840
0.66570	1.00000	1.00000	0.66570
0.92890	1.00000	1.00000	0.92890
0.99910	1.00000	1.00000	0.99910
1.00000	1.00000	1.00000	1.00000

15/15 P[c]	15/15 P[NOTd¦c]	15/15 P[L-NAND]	15/15 P[NOTd and c]
0.00000	1.00000	1.00000	0.00000
0.00000	1.00000	1.00000	0.00000
0.00000	1.00000	1.00000	0.00000
0.00000	1.00000	1.00000	0.00000
0.00000	1.00000	1.00000	0.00000
0.00000	1.00000	1.00000	0.00000
0.00123	1.00000	1.00000	0.00123
0.04239	1.00000	1.00000	0.04239
0.21460	1.00000	1.00000	0.21460
0.73970	1.00000	1.00000	0.73970
1.00000	1.00000	1.00000	1.00000

14/30 P[NOTd and c]	14/30 P[c]	14/30 P[NOTd¦c]	14/30 P[L-NAND]
0.00000	0.00000	1.00000	1.00000
0.00000	0.00000	1.00000	1.00000
0.00000	0.00000	1.00000	1.00000
0.00000	0.00160	0.00000	0.99830
0.00022	0.50070	0.00043	0.49940
0.25558	0.99570	0.25668	0.25980
0.84762	0.99990	0.84770	0.84770
0.99940	1.00000	0.99940	0.99940
1.00000	1.00000	1.00000	1.00000
1.00000	1.00000	1.00000	1.00000
1.00000	1.00000	1.00000	1.00000

Table 11.5: Calculation of P[c], P[$\overline{d}$|c], P[L-NAND], P[$\overline{d}$ and c] for the structures 1/15, 3/15, 5/15, 7/15, 8/15, 9/15, 10/15, 11/15, 13/15, 15/15 and 14/30

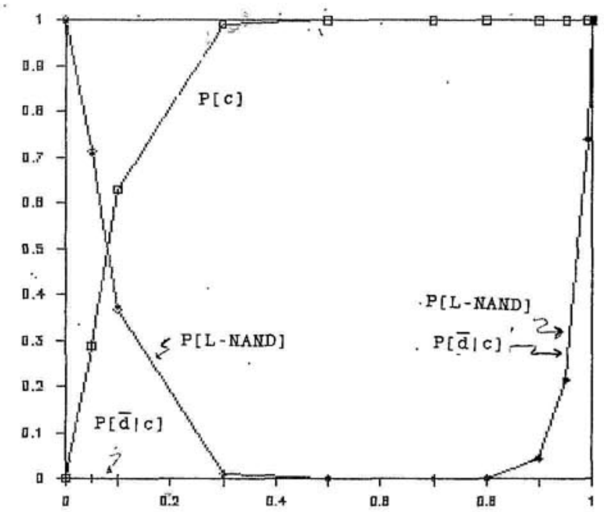


Figure 11.6: The 1/15 structure

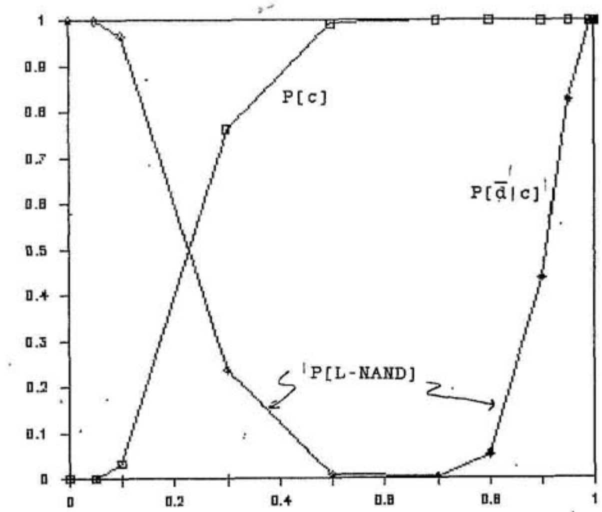


Figure 11.7: The 3/15 structure

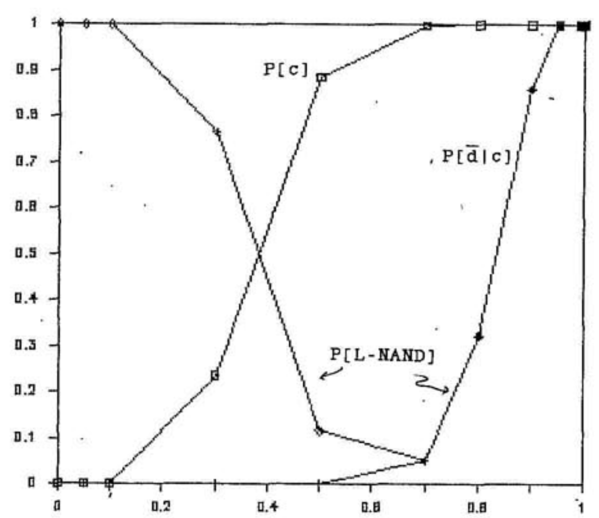


Figure 11.8: The 5/15 structure

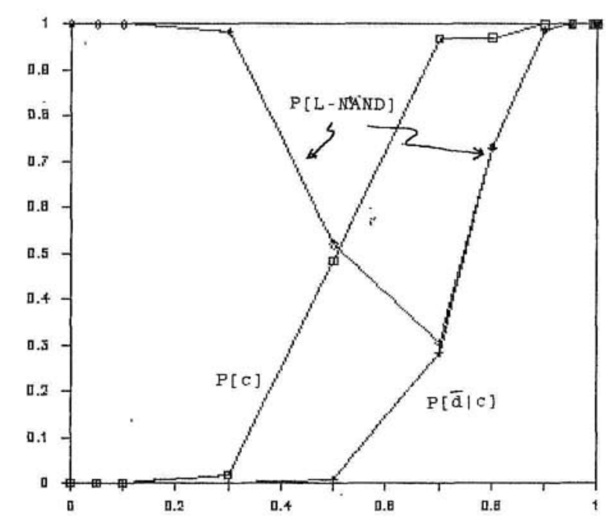


Figure 11.9: The 7/15 structure

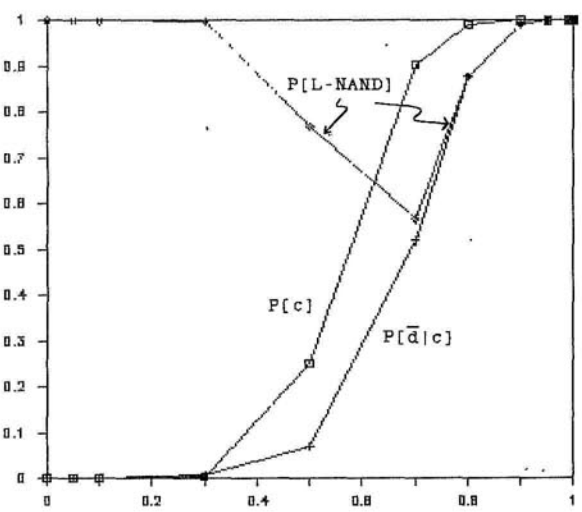


Fig. 11.10: The 8/15 structure

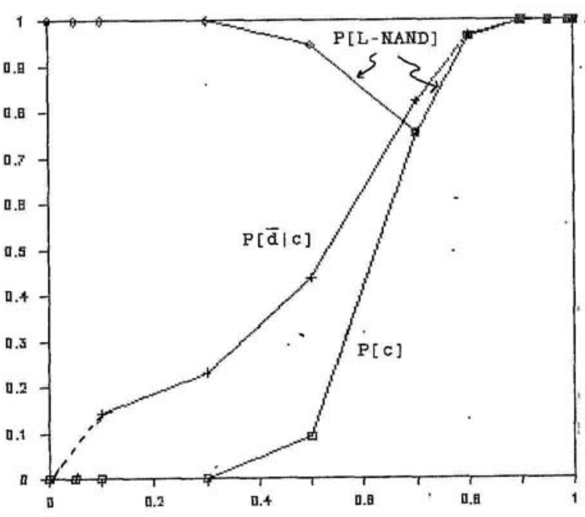


Fig. 11.11: The 9/15 structure

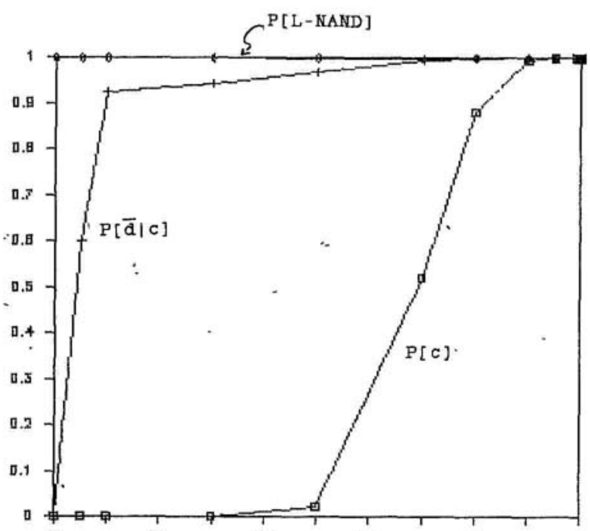


Fig. 11.12: The 10/15 structure

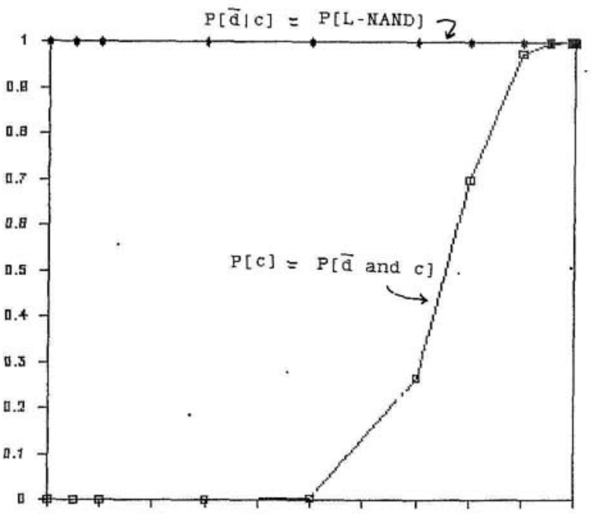
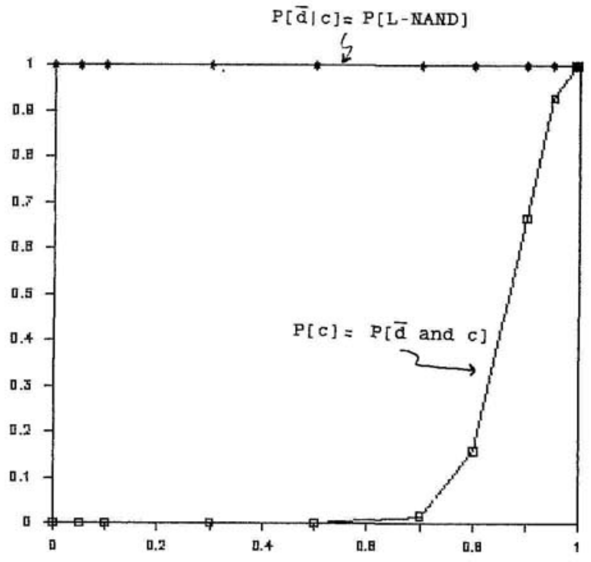


Fig. 11.13: The 11/15 structure



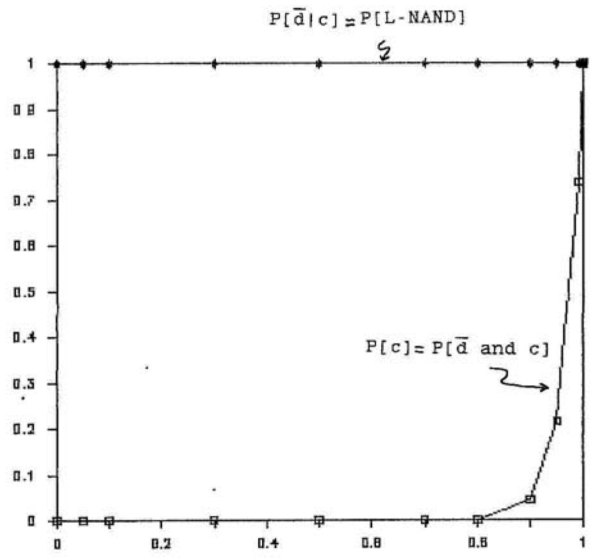


Fig. 11.14: The 13/15 structure

Fig. 11.15: The 15/15 structure

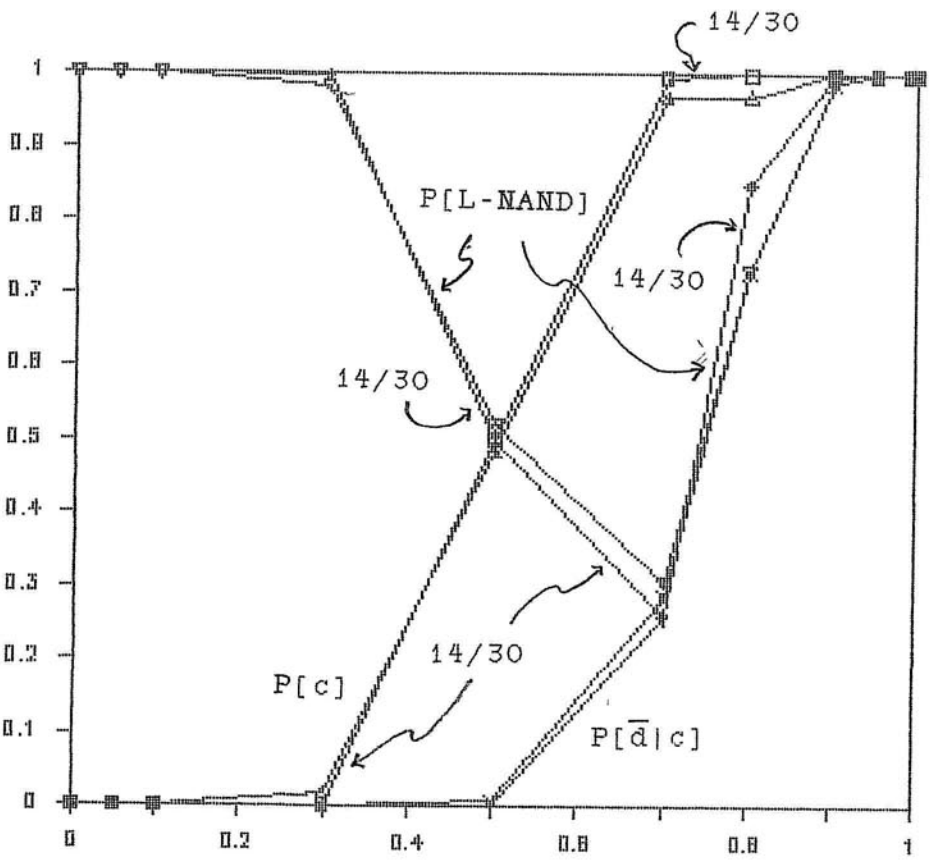


Fig. 11.16: The 7/15 and 14/30 structures

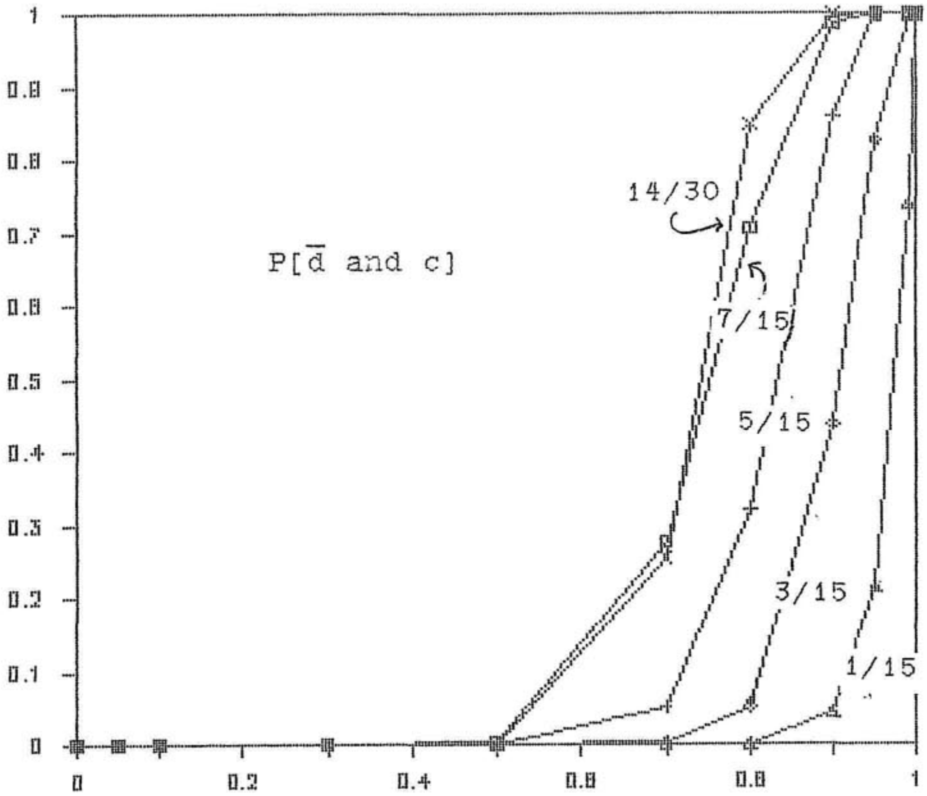


Figure 11.17: A comparison of P[$\overline{d}$ and c] for the structures 1/15, 3/15, 5/15, 7/15, 14/30

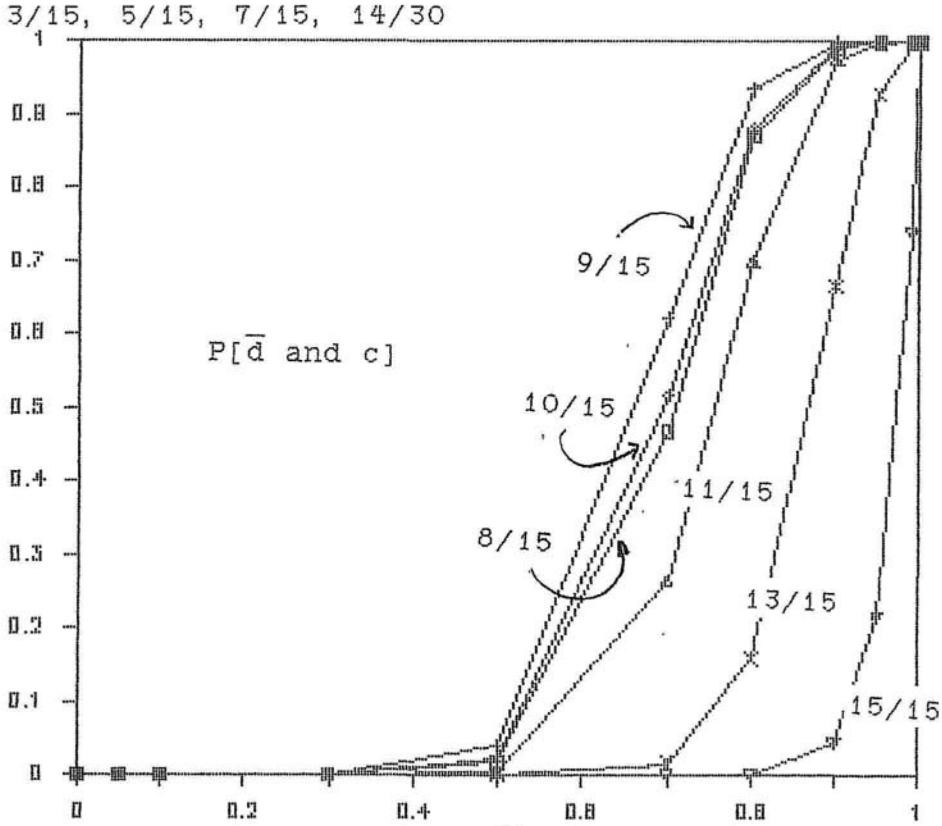


Figure 11.18: A comparison of P[$\overline{d}$ and c] for the structures 8/15, 9/15, 10/15, 11/15, 13/15, 15/15

11.2.2 Restricted Components' Passing Set

The problem now is different. A new approach is needed to calculate the relevant probabilities $P[(a_1,\dots,a_q)\in\hat{T}_{\varphi q}]$ etc. given that the components are NAND consistent and the structure symmetric (k out of n).

Notation:

Let A be the set of statements. By restricting the choice to groups of α statements, we look at all possible such groups ($\binom{A}{\alpha}$ in total) and omit those that are logically inconsistent (i.e. they cannot be chosen by an NAND consistent component). If all the groups are omitted, then the problem is impossible. Suppose now that we represent the remaining groups of α statements, say β in number, as the columns of a matrix **B** with dimensions $A\times\beta$, whose elements $\{b_{ij}\}$ are defined by,

$$b_{ij}=\begin{cases}1 \text{ if statement } a_i \text{ is part of group } j\\ \\ 0 \text{ if statement } a_i \text{ is not part of group } j\end{cases}$$

The jth column of **B** is denoted by $\mathbf{b}_{.j}$ and each column $\mathbf{b}_{.j}$ has exactly α ones.

Let Q be a set of q indices from $\{1,\dots,A\}$. Let also $S(Q)$ be the set of indices from $\{1,\dots,\beta\}$ so that for $j\in S(Q)$ the column $\mathbf{b}_{.j}$ has $b_{ij}=1$ for all $i\in Q$. Define $\mathbf{B}_{S(Q)}$ as the submatrix of **B** which consists of those columns of **B** that belong to $S(Q)$ (say they are γ in number). We assume $q\leq\alpha$ otherwise $S(Q)$ is empty and $\mathbf{B}_{S(Q)}$ does not exist. It is also

possible that even if $q \leq \alpha$ $B_{S(Q)}$ may not exist.

In words, $B_{S(Q)}$ consists of those groups of α statements (columns of **B**) so that each one contains all statements in the chosen index set Q.

Let $\mathbf{B}_{G(Q)}$ be defined as the submatrix of **B** consisting of those columns of **B**, say δ in number, after $\mathbf{B}_{S(Q)}$ has been omitted.

Then, $\mathbf{B}=[\mathbf{B}_{S(Q)}, \mathbf{B}_{G(Q)}]$ where **B** is $A\mathrm{x}\beta$, $\mathbf{B}_{S(Q)}$ is $A\mathrm{x}\gamma$, $\mathbf{B}_{G(Q)}$ is $A\mathrm{x}\delta$.

Let us assume that

$$P[x_i(\mathbf{b}_{.t})=1]=\pi_t \text{ for } i=1,\ldots,n$$

In words, π_t is the probability that the group of α statements represented by column $\mathbf{b}_{.t}$ is accepted by a component, and it is the same for all components.

Let w_i: the number of components that pick the group of α statements represented by column $\mathbf{b}_{.i}$ for $i=1,\ldots,\beta$

Certainly $\Sigma^{\beta}_{i=1}w_i=n$, $w_i \geq 0$ and integer.

Now given a set of q statements $(a_{j(1)},\ldots,a_{j(q)})$ with $Q=\{j_1,\ldots,j_q\}$, we want to determine

$P[(a_{j(1)},\ldots,a_{j(q)})\in\hat{T}_{\varphi q}]=P[\bar{d}$ and $c]$ which, because of NAND consistency of components equals to

$$=P[\varphi^D(\Pi_{j\in Q}\mathbf{x}(a_j))=1 \text{ and } \Pi_{j\in Q}\varphi(\mathbf{x}(a_j))=1]$$

$$=P[\varphi^D(\Pi_{j\in Q}\mathbf{x}(a_j))=1|\Pi_{j\in Q}\varphi(\mathbf{x}(a_j))=1]$$

$$P[\Pi_{j\in Q}\varphi\ (\mathbf{x}(a_j))=1] \qquad (11.51)$$

$$=P[\bar{d}|c]P[c] \qquad (11.52)$$

When φ is a k out of n structure the first term in the RHS of (11.51) is the probability that n-K+1 or more components all accept each of the statements $\{a_j\}$, $j\in Q$, given that for each a_j $(j\in Q)$ there are $k_j \geq k$ components that accept it. But this

requires that n-k+1 or more components choose columns from $B_{S(Q)}$ (n-k+1 matches), given that for each a_j, $j \in Q$ there are $k_j \geq k$ components that accept it.

First we determine the probability that there are r columns exactly chosen from $B_{S(Q)}$. (This is the same as the notion of having r matches in the case of unrestricted component choice set). We will again denote this event by M=r, where M is the random variable denoting the number of columns chosen from $B_{S(Q)}$

$P[M=r]=\Sigma\ (n!/w_1!\ldots w_\beta!)\ \Pi^\beta{}_{t=1}\pi_t{}^{w(t)}$ (11.53)

where the summation is carried over $(w_1,\ldots,w_\beta)\in W_{1Qr}$

where

$W_{1Qr}\equiv\{(w_1,\ldots,w_\beta)\mid\Sigma^\beta{}_{i=1}w_i=n,\ \Sigma_{i\in S(Q)}w_i=r,\ w_i\geq 0$

and integer} (11.54)

W_{1Qr} is the set of all possible voting situations where r votes are casted in favor of columns in *S(Q)*. Alternative ways of expressing P[M=r] are

$P[M=r]=P[\mathbf{w}\in W_{1Qr}\mid\Sigma^\beta{}_{i=1}w_i=n]$

$\quad\quad=P[\Sigma_{i\in S(Q)}w_i=r\mid\Sigma^n{}_{i=1}w_i=n]$

where $\mathbf{w}=(w_1,\ldots,w_\beta)$

Second we will examine the probability that for each a_i with $i\in Q$ there are k_i or more components that accept it: $P[K_j\geq k_j, i\in Q]$. As it is customary in probability theory, we will use K_j to denote the random variable that may take any particular value k_j.

Define $\mathbf{k}_Q$ as a nx1 vector whose components κ_j are given by,

$$\kappa_j = \begin{cases} k_j \text{ if } j \in Q \\ 0 \text{ if } j \notin Q \end{cases}$$

Now define the set of all possible voting situations W_{2Q} that will give at least $\{k_i\}$ votes to statements $\{a_i\}$ $i \in Q$ and perhaps to other statements too,

$$W_{2Q} = \{\mathbf{w} \mid \mathbf{B}\mathbf{w} \geq \mathbf{k}_Q,\ \Sigma^{\beta}{}_{i=1} w_i = n,\ \mathbf{w} \geq 0\} \qquad (11.55)$$

and then using the multinomial distribution ,

$P[K_i \geq k_i, i \in Q] =$

$$= \Sigma\ (n!/w_1! \ldots w_\beta!)\, \pi_1{}^{w(1)} \ldots \pi_\beta{}^{w(\beta)} \qquad (11.56)$$

where the sum is over $\mathbf{w} \in W_{2Q}$.

Note that $P[K_i \geq k_i,\ i \in Q] = P[\mathbf{w} \in W_{2Q} \mid \Sigma^{\beta}{}_{i=1} w_i = n]$.

Third we will calculate the probability that there are k_i or more votes for each a_i, $i \in Q$, given that there are r matches exactly (i.e. there are exactly r components that pick columns from $\mathbf{B}_{S(Q)}$. Namely, $P[K_i \geq k_i; i \in Q$ and $M = r]$.

Define,

$$W_{3Q} = \{\mathbf{w} \mid \mathbf{B}\mathbf{w} \geq \mathbf{k}_Q, \Sigma_{i \in S(Q)} w_i = r, \Sigma^{\beta}{}_{i=1} w_i = n, \mathbf{w} \geq 0\} \qquad (11.57)$$

Observe that $W_{3Q} = W_{2Q} \cap W_{1Qr}$.

Then using the multinomial distribution once more,

$P[K_i \geq k_i,\ i \in Q$ and $M = r] =$

$$= \Sigma\ (n!/w_1! \ldots w_\beta!)\, \pi_1{}^{w(1)} \ldots \pi_\beta{}^{w(\beta)} \qquad (11.58)$$

where the summation is over $\mathbf{w} \in W_{3Q}$.

Note that $P[K_i \geq k_i,\ i \in Q$ and $M = r] = P[\mathbf{w} \in W_{3Q} \mid \Sigma^{n}{}_{i=1} w_i = n]$

Now

$$P[\bar{d} \text{ and } c] = P[M \geq n-k+1 \text{ and } K_i \geq k,\ i \in Q]$$

which is calculated from (11.58)

$P[c]=P[K_i \geq K \text{ for } i \in Q]$

which is calculated from (11.56)

From these quantities we can obtain $P[\bar{d}|c]$ and P[L-NAND] as in the case of unrestricted components passing set.

Example 11.3

Let φ be symmetric (k out of n). Let $\alpha=1$ and consequently $q=1$ $(q \leq \alpha)$. We want to find the probability that at least $n-K+1$ components pick a_1 (say $Q=\{1\}$), given that at least K components pick a_1. This probability is obviously 1 if $K \geq n-K+1$ (or when $\varphi \in \mathcal{S}_2$); while for $K \leq n-K+1$ it equals to h^D/h where

$h^D=\Sigma^n_{r=n-K+1}\binom{n}{r}p_1{}^r(1-p_1)^{n-r}$

$h=\Sigma^n_{r=K}\binom{n}{r}p_1{}^r(1-p_1)^{n-r}$

as we recall from Proposition 11.4 and Remark 1 that follows it; and where p_1 is the probability that a_1 is picked by a component. For reasons of demonstration we will use the theory developed in 11.2.2 to reach the same result.

B: is the identity matrix I (dimensions AxA)

$Q=\{1\}$, $S(Q)=\{1\}$, $\beta=1$

$\mathbf{B}_{S(Q)}$ consists of the first column of **B** only

$\mathbf{B}_{G(Q)}$ is the rest of **B**

Also $w_1=r$ and $\pi_1=p_1$

Take (11.53) which in this case collapses to

$$P[M=r]=\binom{n}{r}p_1{}^r(1-p_1)^{n-r}=\binom{n}{w(1)}p_1{}^{w(1)}(1-p_1)^{n-w(1)} \tag{11.61}$$

From (11.56)

$$P[K_1 \geq K]=P[w_1 \geq K]=\Sigma^n_{w(1)=K}\binom{n}{w(1)}p_1{}^{w(1)}(1-p_1)^{n-w(1)} \tag{11.62}$$

From (11.58)

$$P[K_1 \geq k \text{ and } M=r] = \begin{cases} P[M=r] & \text{if } r \geq k \\ 0 & \text{if } r<k \end{cases}$$

Thus,

$$P[M=r \mid K_1 \geq k] = \begin{cases} P[M=r]/P[K_1 \geq k] & \text{if } r \geq k \\ 0 & \text{if } r<k \end{cases}$$

Then,

$$P[M \geq n-k+1 \mid K_1 \geq k] = \frac{\Sigma^n_{r=\max(k,\, n-k+1)} \binom{n}{r} p_1^{\,r} (1-p_1)^{n-r}}{\Sigma^n_{r=k} \binom{n}{r} p_1^{\,r} (1-p_1)^{n-r}} \tag{11.63}$$

which equals to 1 when $k \geq n-k+1$ as expected since then $\varphi \in \mathcal{S}_2$.

Example 11.4

Let φ be symmetric k out of n. Let $\alpha=2$ and $q=1$ $(q \leq \alpha)$ and $A=3$. Suppose all combinations $\binom{A}{\alpha}$ are consistent thus $\binom{3}{2}=3$ groups of statements are possible. Now **B** becomes,

$$\mathbf{B} = \begin{array}{c} \begin{array}{ccc} \mathbf{b}_{.1} & \mathbf{b}_{.2} & \mathbf{b}_{.3} \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \end{array}$$

Let $Q=\{1\}$. We want to find if a_1 gets n-k+1 or more votes, given it gets k votes or more. Observe that

$\mathcal{S}(Q)=\{1\}$ (column $\mathbf{b}_{.1}$)

$\mathbf{B}_{S(Q)}$ consists of column $\mathbf{b}_{.1}$

$\mathbf{B}_{G(Q)}=[\mathbf{b}_{.2}, \mathbf{b}_{.3}]$

Also $w_1+w_2+w_3=n$ while π_i is the probability that a component picks $b_{.i}$ $(i=1,2,3)$. Certainly,

$$P[M=r]=\binom{n}{r}\pi_1^{r}(1-\pi_1)^{n-r} \tag{11.64}$$

Look at W_{2Q}. $Bw\geq K_Q$ is now,

$$\begin{aligned} w_1 \quad\quad &+w_3\geq k \\ w_1+w_2 \quad &\geq 0 \\ w_2&+w_3\geq 0 \end{aligned}$$

The solution to this system requires that

(a) When $w_1\leq k-1$ then $w_3\geq k-w_1$ and $w_2+w_3=n-w_1$

(b) When $w_1\geq k$ then we only need $w_2+w_3=n-w_1$

Thus

$P[K_1\geq k]=P[c]=$

$$\sum_{w_1=0}^{k-1}\ \sum_{w_3=k-w_1}^{n}(n!/w_1!w_3!(n-w_1-w_3)!)\pi_1^{w(1)}\pi_3^{w(3)}$$

$$(1-\pi_1-\pi_3)^{n-w(1)-w(3)}+$$

$$+\sum_{w_1=k}^{n}\binom{n}{w(1)}\pi_1^{w(1)}(1-\pi_1)^{(n-w(1)} \tag{11.65}$$

Also

$P[K_1\geq k \text{ and } M=r]=P[w_3\geq k-r \text{ and } w_1=r\,|\,w_1+w_2+w_3=n]=$

$$=\binom{n}{r}\pi_1^{r}\sum_{w(3)=k-r}^{n-r}\binom{n-r}{w(3)}\pi_3^{w(3)}\pi_2^{n-r-w(3)} \tag{11.66}$$

From (11.66) we can calculate $P[\overline{d}$ and $c]=P[K_1\geq k$ and $M\geq n-k+1]$ by summing over r from n-k+1 to n. Hence the P[L-NAND] of the structure can also be determined.

Remark 1

As in the case where components are not restricted to choose exactly α statements from A, we may wish to evaluate an "average" or the expected value of $P[\overline{d}$ and $c]$ of a k/n structure as each π_i varies over $(0, 1)$ uniformly:

$E\{P[\overline{d}$ and $c]\}=$

$$=\Sigma(n!/(w_1!\ldots w_\beta!))$$

$$\int_{\mathcal{R}} u_1^{w(1)}\ldots u_\beta^{w(\beta)}du_1\ldots du_\beta$$

where

(a) The summation is over all $\mathbf{w}$ such that

$\mathbf{Bw}\geq ke$, $\Sigma_{i\in S(Q)}w_i\geq n-k+1$, $\Sigma^\beta_{i=1}w_i=n$, $\mathbf{w}_i\geq 0$ and integer for all $i=1,\ldots,\beta$.

(b) The region $\mathcal{R}$ over which the integration is carried is defined by all u_i so that $\Sigma^\beta_{i=1}u_i=1$.

The integral over $\mathcal{R}$ has the form of Dirichlet integral (a generalized form of the beta function) and can be evaluated in closed form. It is possible therefore to calculate the average performance of the k/n structure for a group of q statements both for giving answers and respecting logic for the case of restricted choice of components

Remark 2

Monte Carlo techniques can be used to generate voting situations and then calculate the relevant probabilities of interest for types of consistencies for a k/n structure. For example generate $\mathbf{w}=w_1,\ldots,w_\beta$ with $\Sigma w_i=n$ and check if $\mathbf{w}\in W_{3Q}$.

Remark 3

A necessary condition for the non emptiness of W_{2Q} can be found by summing the inequalities in $\mathbf{Bw} \geq k_Q$ and noting that $\Sigma w_i = n$. Namely, $\alpha n \geq \Sigma_{i \in Q} k_i$ and since $k_i \geq k$,

$\alpha n \geq |Q| k$.

11.3 Symmetric Structures with Components that May not Be Logically Consistent

Let a structure φ be K/n and suppose for simplicity that the set of statements A contains two statements a_1, a_2 and the negation of their conjunction. For more than two statements the ideas that are exposed below remain the same, only the formulas become more involved.

Suppose that,

p_0: the probability that a component i (for all i) rejects $NOT(a_1 \wedge a_2)$ given that he rejects both a_1 and a_2:

$$=P[x_i(NOT(a_1 \wedge a_2))=0 \mid x_i(a_1)=0,\ x_i(a_2)=0]$$

p_1: the probability that a component i rejects $NOT(a_1 \wedge a_2)$ given that he accepts one of the statements a_1, a_2 and rejects the other.

$$=P[x_i(NOT(a_1 \wedge a_2))=0 \mid x_i(a_1)=0,\ x_i(a_2)=1]$$

$$=P[x_i(NOT(a_1 \wedge a_2))=0 \mid x_i(a_1)=1,\ x_i(a_2)=0]$$

p_{12}=probability that a component i rejects $NOT(a_1 \wedge a_2)$ given that he accepts both a_1 and a_2

$$=P[x_i(NOT(a_1 \wedge a_2))=0 \mid x_i(a_1)\,x_i(a_2)=1]$$

Observe that when component i is L-NAND then $(x_i(a_1)=1,\ x_i(a_2)=1) \Rightarrow x_i(NOT(a_1 \wedge a_2))=0$ and hence $p_{12}=0$. Also if i is R-NAND $x_i(NOT(a_1 \wedge a_2))=0 \Rightarrow x_i(a_1)=1,\ x_i(a_2)=1$ and hence $p_1=0$ and $p_0=0$. Naturally, if i is NAND then $p_{12}=1$, $p_1=0$, $p_0=0$.

We want to calculate

$P[(a_1, a_2) \in \hat{T}_{\varphi 2}]=$

$$=P[\varphi(\mathbf{x}(\mathrm{NOT}(a_1 \wedge a_2))=0 \mid \varphi(\mathbf{x}(a_1))\varphi(\mathbf{x}(a_2))=1]$$
$$P[\varphi(\mathbf{x}(a_1))\varphi(\mathbf{x}(a_2))=1] \quad (11.68)$$

If we allow the following symbolism,

R_2= the random variable (r.v.) representing the number of components that reject NOT$(a_1 \wedge a_2)$. Then r_2 represents any particular number R_2 may take.

K_1= the r.v. representing the number of components that accept statement a_1. Then k_1 is any particular realization of K_1.

K_2= the r.v. representing the number of components that accept statement a_2. Then k_2 is any particular realization of K_2.

M_2= the r.v. representing the number of components that accept both a_1 and a_2 individually. Then μ_2 is any particular realization of M_2.

Then (11.68) becomes,

$P[\varphi(\mathbf{x}(\mathrm{NOT}(a_1 \wedge a_2))=0 \text{ and } \varphi(\mathbf{x}(a_1))\varphi(\mathbf{x}(a_2))=1]=$

$$=\Sigma^{n}{}_{k(1),k(2)\geq k}\, P[R_2 \geq n-k+1 \text{ and } K_1=k_1, K_2=k_2] \qquad (11.69)$$

But

$P[R_2 \geq n-k+1 \text{ and } K_1=k_1,\ K_2=k_2]=$

$$=P[R_2 \geq n-k+1 \mid K_1=k_1,\ K_2=k_2]P[K_1=k_1, K_2=k_2] \qquad (11.70)$$

Now if we condition on M_2,

$$=P[R_2 \geq n-K+1 \mid M_2=\mu_2, K_1=k_1, K_2=k_2]\,P[M_2=\mu_2 \mid K_1=k_1, K_2=k_2]$$
$$P[K_1=k_1, K_2=k_2] \qquad (11.71)$$

Now,

$$P[R_2=r_2 \mid M_2=\mu_2, K_1=k_1, K_2=k_2]=$$
$$=\Sigma\ \mathrm{Bin}(n-k_1-k_2+\mu_2, u_0, p_0)\ \mathrm{Bin}(k_1+k_2-2\mu_2, u_1, p_1)$$
$$\mathrm{Bin}(\mu_2, u_{12}, p_{12}) \qquad (11.72)$$

where Bin(N, K, p) is the binomial distribution i.e. the probability that there are K successes from a population of N trials when the probability of success is p. The summation in (11.72) is carried over all possible u_0, u_1, u_{12} such that $u_0+u_1+u_{12}=r_2$.
From (11.72) we obtain $P[R_2 \geq n-K+1 \mid M_2=\mu_2, K_1=k_1, K_2=k_2]$ by summing over r_2 from n-K+1 to n.
Finally, $P[M_2=\mu_2 \mid K_1=k_1, K_2=k_2]$ is given by the hypergeometric distribution as we know from (11.39). While, $P[K_1=k_1, K_2=k_2]$ is given by the product of two binomial distributions as in (11.44). Thus, we may substitute back to (11.71) and then summing over k_1 and k_2 from k to n as is required by (11.69), we obtain the LHS of (11.69) which is also known by the name $P[\overline{d}$ and c].

We can check that when the components are NAND consistent, the RHS of (11.72) collapses to zero unless $r_2=\mu_2$ and then only one term survives, namely, when $u_1=0$, $u_2=0$, $u_3=r_2=\mu_2$ and thus the RHS of (11.72) becomes equal to 1. Then the formulas become identical to those found in section 11.1 as expected.

Chapter 12

RELATION TO OTHER THEORIES

12.1 Relation to Reliability Theory

Given a symmetric structure composed of components that may be ON or OFF with some probability, reliability theory is concerned with determining the probabilistic behaviour of the structure as a whole. The similarity with our problem is obvious. In our case components are ON or OFF when presented with an issue a from a set of alternatives A. Indeed, our aim was to determine,

$$P[\varphi(\mathbf{x}(a|A))=1]=E\varphi(\mathbf{x}(a|A))$$

If a and A are suppressed in the the notation $\mathbf{x}(a|A)$ and if we let $p_i(a|A)=p_i=P[x_i(a|A)=1]$ for $i=1,...,n$, then assuming that the components are independent (i.e. $x_1,...,x_n$ are independent random variables) we can denote,

$$h(\mathbf{p})=E\varphi(\mathbf{x}) \text{ where } \mathbf{p}=p_1,...,p_n$$

and $h(\mathbf{p})$ is called the *reliability function* because it represents the probability that the structure (or system) is in operating condition (ON).

Reliability theory, among other things, covers the issue of calculating $E\varphi(\mathbf{x})$. Some results are shortly presented here. The interested reader will find a nice exposition of reliability theory in the book by R.E. Barlow and F. Proschan [1975].

The straightforward way to compute the reliability function of a structure φ when its components are independent, is to expand $\varphi(\mathbf{x})$, which is a multilinear function of the x_i's, and then take expectation, thus, replacing each x_i by the corresponding p_i.

To calculate (expand) $\varphi(\mathbf{x})$ one uses the fact that it can be expressed as a parallel structure of min paths,

$$\varphi(\mathbf{x})=\sqcup^{m}{}_{j=1}\ \Pi_{i\in P(j)}\ x_i \tag{12.1}$$

where $P_1,\ldots,P_m$ are the min paths of φ;

or as a series structure of min cuts,

$$\varphi(\mathbf{x})=\Pi^{l}{}_{j=1}\ \sqcup_{i\in C(l)}\ x_i \tag{12.2}$$

where $C_1,\ldots,C_l$ are the min cuts of φ.

The calculation is usually formidable and therefore theorems on bounds of $h(\mathbf{p})$ are used like,

$$\Pi^{l}{}_{j=1}\ \sqcup_{i\in C(j)}\ p_i \leq h(\mathbf{p}) \leq \sqcup^{m}{}_{j=1}\ \Pi_{i\in P(j)}\ p_i \tag{12.3}$$

For a k/n structure $h(\mathbf{p})$ takes the form

$$h(\mathbf{p})=\Sigma^{n}{}_{l=k}\ \Sigma\ p_{i(1)}\cdots p_{i(l)}(1-p_{j(1)})\cdots(1-p_{(j(n-l)})) \tag{12.4}$$

where the inner summation is carried over all sets of l indices $\{i_1,\ldots,i_l\}\in\{1,\ldots,n\}$ and $\{j_1,\ldots,j_{n-l}\}=\{1,\ldots,n\}-\{i_1,\ldots,i_l\}$. If in particular $p_1=p_2=\ldots=p_n=p$ then for a k/n structure

$$h(\mathbf{p})=h(p)=\Sigma^{n}{}_{l=k}\ ({}^{n}{}_{l})\,p^{l}(1-p)^{n-l} \tag{12.5}$$

It is known that $h(\mathbf{p})$ for any coherent structure where $p_1=\ldots=p_n$ is s-shaped (i.e. it is nondecreasing in p and it crosses the 45 degree line once only from below).

It is also known that for a k/n structure h(p), there is a $0\leq p^*\leq 1$ so that h(p) is convex on $0\leq p\leq p^*$ and concave on $p^*\leq p\leq 1$. A proof of this appears later in Proposition 12.2

As $h(\mathbf{p})$ is multilinear in the p_i's, if we take partial derivatives we will obtain,

$$\frac{\partial}{\partial p_i}h(\mathbf{p})=h(1_i,\mathbf{p})-h(0_i,\mathbf{p}) \tag{12.6}$$

where

$$h(1_i,\mathbf{p})=h(p_1,\ldots,p_{i-1},1,p_{i+1},\ldots,p_n)$$

$$h(0_i,\mathbf{p})=h(p_1,\ldots,p_{i-1},0,p_{i+1},\ldots,p_n)$$

The proof of (12.6) uses the fact that

$$h(\mathbf{p})=p_i h(1_i,\mathbf{p})+(1-p_i)h(0_i,\mathbf{p})$$

Similarly,

$$\frac{\partial^2}{\partial p_i \partial p_j}h(\mathbf{p})=h(1_i,1_j,\mathbf{p})-h(1_i,0_j,\mathbf{p})-h(0_i,1_j,\mathbf{p})+h(0_i,0_j,\mathbf{p}) \tag{12.7}$$

Proposition 12.1

Let $h(\mathbf{p})$ be the reliability function of a coherent structure $\varphi(\mathbf{x})$. Also let $h_{ij}(\mathbf{p})$ be the reliability function of the contraction of $\varphi(\mathbf{x})$ when we set $x_i=x_j$, then,

$$h(\mathbf{p})-h_{ij}(\mathbf{p})=p(1-p)\frac{\partial^2}{\partial p_i \partial p_j}h(\mathbf{p}) \tag{12.8}$$

$$=p(1-p)\{h(1_i,0j,\mathbf{p})+h(0_i,1_j,\mathbf{p})-h(1_i,1_j,\mathbf{p})-h(0_i,0_j,\mathbf{p})\} \tag{12.9}$$

where $p=p_i=p_j$.

Proof:

Any coherent structure $\varphi(\mathbf{x})$ can be expanded to,

$$\varphi(\mathbf{x})=x_i x_j \varphi(1_i,1_j,\mathbf{x})+x_i(1-x_j)\varphi(1_i,0_j,\mathbf{x})+$$

$$+(1-x_i)x_j\varphi(0_i,1_j,\mathbf{x})+(1-x_i)(1-x_j)\varphi(0_i,0_j,\mathbf{x}) \tag{12.10}$$

The contracted structure $\varphi_{ij}(\mathbf{x})$ where x_i is set equal to x_j will satisfy,

$$\varphi_{ij}(\mathbf{x})=x_i\varphi(1_i,1_j,\mathbf{x})+(1-x_i)\varphi(0_i,0_j,\mathbf{x}) \tag{12.11}$$

It follows that,

$$h(\mathbf{p})-h_{ij}(\mathbf{p})=E\varphi(\mathbf{x})-E\varphi_{ij}(\mathbf{x})=$$

$$=p(1-p)\{h(1_i,0_j,\mathbf{p})+h(0_i,1_j,\mathbf{p})-h(1_i,1_j,\mathbf{p})-h(0_i,0_j,\mathbf{p})\}$$

and because of (12.7),

$$=p(1-p)\frac{\partial^2}{\partial p_i \partial p_j}h(\mathbf{p}) \qquad //$$

Let $\varphi(\mathbf{x})$ be a n component coherent structure. We will say that $\varphi(1_i,\mathbf{x})$ is a *fixing* of the structure $\varphi(\mathbf{x})$. The name is justified since component i is fixed so that $x_i=1$. Certainly, $\varphi(1_i,\mathbf{x})$ is a n-1 component structure. The same holds for $\varphi(0_i,\mathbf{x})$ where component i is fixed so that $x_i=0$ while the rest of the components (n-1) can vary and are denoted as $\mathbf{x}$.

In general if $\varphi(\mathbf{x})$ is a n component structure then $\varphi(1_{i(1)},1_{i(2)},\dots,1_{i(I)},0_{j(1)},\dots,0_{j(J)},\mathbf{x})$ will also be called a *fixing* of φ where components $x_{i(1)}=\dots=x_{i(I)}=1$ and $x_{j(1)}=\dots=x_{j(J)}=0$ while the rest of the components, n-I-J, are free to vary and retain the notation $\mathbf{x}$. Certainly, the new structure is a n-I-J component structure.

In particular, when $\varphi(\mathbf{x})$ is a symmetric K/n structure,

(a) $\varphi(1_i,\mathbf{x})$ is a K-1/n-1 structure.

This holds, because all paths that do not contain x_i will be redundant in the presence of paths that contain x_i before it was fixed to be equal to 1.

(b) $\varphi(0_i,\mathbf{x})$ is a K/n-1 structure.

This holds because all paths that contain x_i in $\varphi(\mathbf{x})$ will be omitted now, while the rest of the paths are all possible com-

binations of n-1 components taken k at a time.

In general, if $\varphi(\mathbf{x})$ is a k/n structure and if I components are fixed to be equal to 1, while J components are fixed to be equal to 0, then the resulting structure is a k-I/n-I-J symmetric structure.

Based now on the notion of fixing we can prove the following,

Proposition 12,2

Let $h(\mathbf{p})$ be the reliability function of a symmetric k/n structure then,

(a)

$$\frac{\partial}{\partial p_i} h(\mathbf{p}) \Big|_{\mathbf{p}=p\mathbf{e}} = h(1_i, \mathbf{p}) \Big|_{\mathbf{p}=p\mathbf{e}} = \binom{n-1}{k-1} p^{k-1} (1-p)^{n-k} \qquad (12.12)$$

where $\mathbf{e}=(1,\ldots,1)$, a vector of n ones.

(b)

$$\frac{d}{dp}\left[h(\mathbf{p}) \Big|_{\mathbf{p}=p\mathbf{e}}\right] = \left[\sum_i \frac{\partial}{\partial p_i} h(\mathbf{p})\right]_{\mathbf{p}=p\mathbf{e}} = \binom{n}{k} k p^{k-1} (1-p)^{n-k} \qquad (12.13)$$

(c)

$$\frac{\partial^2}{\partial p_i \partial p_j} h(\mathbf{p}) \Big|_{\mathbf{p}=p\mathbf{e}} = \left[h(1_i, 1_j, \mathbf{p}) + h(0_i, 0_j, \mathbf{p}) - h(1_i, 0_j, \mathbf{p}) - h(0_i, 1_j, \mathbf{p})\right]_{\mathbf{p}=p\mathbf{e}}$$

$$= \frac{1}{k-1} \binom{n-2}{k-2} p^{k-1} (1-p)^{n-k-1} [k-1-(n-1)p] \qquad (12.14)$$

(d)

$$\frac{d^2}{dp^2}[h(\mathbf{p})\Big|_{\mathbf{p}=p\mathbf{e}}] = [\sum_{i,j} \frac{\partial^2}{\partial p_i \partial p_j} h(\mathbf{p})\Big|_{\mathbf{p}=p\mathbf{e}}]$$

$$= \binom{n}{k} k p^{k-2} (1-p)^{n-k-1} [k-1-(n-1)p] \qquad (12.15)$$

Proof:

(a) Because of (12.6)

$$\frac{\partial}{\partial p_i} h(\mathbf{p}) = h(1_i, \mathbf{p}) - h(0_i, \mathbf{p})$$

Using now the fact that $h(1_i, \mathbf{p})$ is k-1/n-1, $h(0_i, \mathbf{p})$ is k/n-1 while $\mathbf{p}$ is set to be $p\mathbf{e}$, we obtain,

$$\frac{\partial}{\partial p_i} h(\mathbf{p})\Big|_{\mathbf{p}=p\mathbf{e}} =$$

$$= \Sigma^{n-1}{}_{j=k-1} p^j (1-p)^{n-j} - \Sigma^{n-1}{}_{j=k} p^j (1-p)^{n-j} =$$

$$= \binom{n-1}{k-1} p^{k-1} (1-p)^{n-k}$$

(b) $$\frac{d}{dp}[h(\mathbf{p})\Big|_{\mathbf{p}=p\mathbf{e}}] = \sum_{i=1}^{n} \frac{\partial}{\partial p_i} h(\mathbf{p}) \frac{\partial p_i}{\partial p}\Big|_{p_i=p} = \sum_{i=1}^{n} \frac{\partial}{\partial p_i} h(\mathbf{p})\Big|_{\mathbf{p}=p\mathbf{e}}$$

and use (12.12) to complete the proof.

(c) Use (12.7) and the fact that a fixing of a k/n structure is another symmetric structure.

(d) Sum the terms using (12.14) //

From Proposition 12.2, it follows that the reliability function $h(p)$ of a k/n structure is convex on $[0, p^*]$ and concave on $[p^*, 1]$ where $p^* = (k-1)/(n-1)$. The convex-concave form of the reliability function of k/n structures has been shown by Bartozinsky [1972]

in the context of social choice theory.

The property of convex-concave reliability function can be shown to hold also for simple two component contractions of K/n structures:

Proposition 12.3

Let K/n have a reliability function $h(\mathbf{p})$ and a two component $\{i,j\}$ contraction of K/n have a reliability function $h_{ij}(\mathbf{p})$. Then,

$$n^2[h(\mathbf{p})-h_{ij}(\mathbf{p})\Big|_{\mathbf{p}=p\mathbf{e}}] = p(1-p)\frac{d^2}{dp^2}[h(\mathbf{p})\Big|_{\mathbf{p}=p\mathbf{e}}] \tag{12.16}$$

Proof:

Summing (12.8) over all possible values of i, j (n^2 in total) and since $h_{ij}(\mathbf{p})$ is the same for all choices of (i, j), (12.16) is obtained. //

Because of Proposition (12.2) we know that $h(\mathbf{p})$ is convex-concave with inflection point at $(K-1)/(n-1)$. This means that the RHS of (12.16) is first positive and then negative. It follows that $h_{ij}(\mathbf{p})$ crosses $h(\mathbf{p})$ along the ray $\mathbf{p}=p\mathbf{e}$ at its inflection point $p^*=(K-1)/(n-1)$, and that it lies above h(p) for $p\leq p^*$ and below h(p) for $p\geq p^*$. Further, the reliability function $h_{ij}(p)$ satisfies the following,

Proposition 12.4

The reliability function $h_{ij}(\mathbf{p})$ of a two component $\{i,j\}$ contraction of a K/n structure $h(\mathbf{p})$, is convex-concave along the ray

$\mathbf{p}$=pe with a single inflection point at $p^*=(K-1)/(n-1)$.

Proof:

$$h_{ij}(\mathbf{p})=p_{ij}h(1_i,1_j,\mathbf{p})+(1-p_{ij})h(0_i,0_j,\mathbf{p}) \tag{12.17}$$

Let $p_{ij}=p$, $\mathbf{p}$=pe then (12.17) becomes,

$$\frac{d^2}{dp^2}h_{ij}(p)=\frac{d^2}{dp^2}ph(1_i,1_j,p)+\frac{d^2}{dp^2}(1-p)h(0_i,0_j,p) \tag{12.18}$$

Using now the fact that $h(1_i,1_j,p)$ is K-2/n-2 and $h(0_i,0_j,p)$ is K/n-2 we can apply Proposition 12.2 to calculate the respective derivatives. After some algebraic manipulations one observes that (12.18) changes sign once from positive to negative as p varies from 0 to 1 at $p^*=(K-1)/(n-1)$. //

The following Figure 12.1 summarizes Propositions 12.2, 12.3 and 12.4. It shows that both h(p) and $h_{ij}(p)$ are convex-concave with a single inflection point at $p^*=(K-1)/(n-1)$ when h(p) is K/n. Further, $h_{ij}(p)$ lies above h(p) on $(0,p^*)$ and below on $(p^*,1)$.

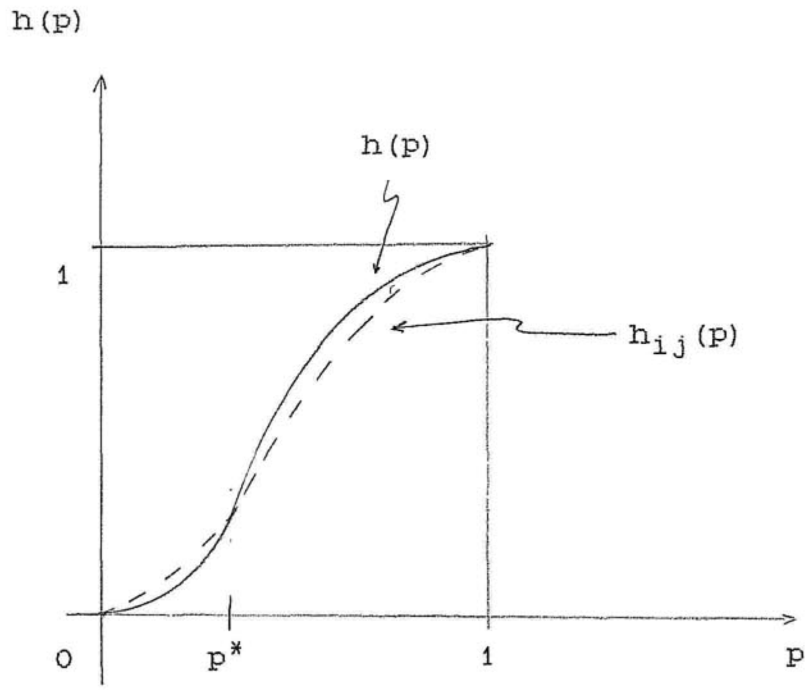


Figure 12. 1: h(p) and $h_{ij}(p)$ when h(p) is k/n.

12.2 Relation to Utility Theory

The probability that a component i accepts a statement a, indirectly implies a cardinal ordering of the statements in the set A. It is therefore natural to ask if there is any relation to utility theory. In fact it is demonstrated that these probabilities can be considered to represent utilities.

12.2.1 Alternatives in A are not random variables.

When alternatives in A are not random variables (lotteries in utility theory terminology) we can define a function $v_i(a|A)$ for component i, so that,

$$v_i(a|A) \equiv P[X_i(a|A)=1]=E[X_i(a|A)] \qquad (12.23)$$

where we use $X_i(a|A)$ instead of $x_i(a|A)$ to underline the fact (as is customary in probability theory) that $X_i(a|A)$ is a random variable. In our problem, as usual, $X_i(a|A)$ takes the values 1 or 0.

Then $v_i(a|A)$ implies a cardinal ordering on the alternatives in A and can be thought of as expressing the preference ordering of component i when presented with a set of alternatives A. In particular, $v_i(a|A)$ expresses the probability or frequency with which component i accepts a when presented with the set of statements A.

If the choice set A is not known but alternatives are

presented to component i for acceptance or rejection, then we may think that component i visualizes all hypothetical situations and thus sets of alternatives within which any particular statement presented to him may arise and then takes an expectation, according to his subjective probabilities that each such hypothetical situation may realize, over possible sets A. In symbols,

$$v_i(a) \equiv E_A P[X_i(a|A)=1] = E_A v_i(a|A) \qquad (12.14)$$

Still another assignment of probabilities to statements of a set A arises when component i is restricted to choose a fixed number of statements from within a set A. In short component i is assumed to have the ability of assessing probabilities to any type of situation that may arise; thus calculating expectations and finally assigning probabilities of being accepted to each statement presented to him. Certainly, these orderings will change drastically if restrictions in the environment or choice set are imposed. But after all, isn't it natural?

12.2.2 Alternatives in A Are Random Variables

When the statements in A are random variables we will first define the conditional probability that component i accepts the statement (lottery) $\mathfrak{a} \in A$ ($\mathfrak{a}$ is a random variable) given that $\mathfrak{a}=a$, where a is a particular realization of the random variable $\mathfrak{a}$,

$$w_i(a|A) \equiv P[X_i(\mathfrak{a}|A)=1|\mathfrak{a}=a] =$$
$$= E_{X(i)|\mathfrak{a}}\, X_i(\mathfrak{a}|A)$$

In other words, $w_i(a|A)$ expresses the probability component i will prefer the fixed amount (or certain outcome) a to the lot-

teries in $A-\{\mathfrak{a}\}$

But, in fact we are interested in determining the probability with which component i will choose statement (or lottery) $\mathfrak{a}$ from the set A,

$$v_i(\mathfrak{a}|A) \equiv P[X_i(\mathfrak{a}|A)=1] = E_{X(i)} X_i(\mathfrak{a}|A) \quad (12.25)$$

but because of probability calculus,

$$v_i(\mathfrak{a}|A) = E_{\mathfrak{a}}[E_{X(i)|\mathfrak{a}}[X_i(\mathfrak{a}|A)]]$$
$$= E_{\mathfrak{a}}[P[X_i(\mathfrak{a}|A)=1|\mathfrak{a}=a]]$$
$$= E_{\mathfrak{a}} w_i(a|A) \qquad (12.26)$$

Observe that $w_i(a|A)$ assigns a value to each outcome a of the random variable $\mathfrak{a}$, which belongs to the set A, and therefore, it can be considered to be a utility function of component i in view of set A.

Under this interpretation it follows that $v_i(\mathfrak{a}|A)$ is the expected utility (associated with the above utility function) of component i for statement (lottery) $\mathfrak{a}$ when his choice is limited to within the set of lotteries in A.

If the set A is not specifically described but a lottery $\mathfrak{a}$ is presented to component i to be accepted or rejected, we will need to define $w_i(a)$,

$$w_i(a) = E_A w_i(a|A) =$$
$$= E_A[P[X_i(\mathfrak{a}|A)=1|\mathfrak{a}=a]]$$

(12.27)

and also

$$v_i(\mathfrak{a}) = E_A v_i(\mathfrak{a}|A)$$
$$= E_A E_{\mathfrak{a}} w_i(a|A)$$
$$= E_{\mathfrak{a}} E_A w_i(a|A)$$
$$= E_{\mathfrak{a}} w_i(a) \qquad (12.28)$$

The quantities $w_i(a)$, $v_i(\mathfrak{a})$ (without reference to a par-

ticular set of alternatives A) are closer to the usual notion of being a utility function and the expected utility of component i respectively. In particular, $w_i(a)$ can be associated to the usual notion of utility of money as it represents the probability that component i will accept the fixed amount a to any possible lottery that might arise in an uncertain environment that randomly presents possible sets of alternative lotteries A.

12.3 Relation to Multilinear Utility Functions, Cardinal Ordering and Arrow Conditions

Before the introduction of probabilities at the component level, we were concerned only with "1 or 0" answers of the components to statements. The answers of the structure, in this case, may or may not be consistent, depending on the form of the structure, logic complexity of the statements, restrictions on A and the number of statements each component accepts, as well as the logic consistency of components. These considerations were developed in preceding chapters and they are related to the fact that Arrow's conditions for aggregating *ordinal* preferences cannot all be satisfied. Ordinal preferences, of course, are related to accepting or rejecting statements (1 or 0 answers of components). To see this consider for example the following statements that induce ordinal preference among $b_1, b_2, b_3 \ldots$

Statement a_1: "b_1 is preferred to b_2"

Statement a_2: "b_3 is preferred to b_5"

etc.

However, when probabilities are introduced at the component level, a cardinal ordering of statements is implied by each component. In this case, if components decide independently, the reliability function $h(\mathbf{p}(a|A))$ of the structure φ, which represents the probability that the structure will accept statement a given the set of alternatives A, is linear in each $p_i(a|A)$ and it obeys the conditions set by Arrow (paraphrased for cardinal ordering) for aggregating individual preferences.

The condition on irrelevant alternatives will be violated at the structure level whenever the components do not obey it (which is quite reasonable). However, when component's i choice does not depend on A nor is it restricted to accept a specified number of statements, then the condition on irrelevant alternatives holds at the component level and also at the structure level.

The linearity of $h(\mathbf{p}(a|A))$ in each $p_i(a|A)$ along with the interpretation of $p_i(a|A)$ as a utility function that was exposed in section 12.2, reminds us of utility functions with "multiple objectives" as described by R. Keeney [1972], R. Keeney and Kirkwood [1975] etc. It is interesting that we arrive at these aggregation forms starting from real life decision structures such as coherent structures, k out of n structures etc. Keeney also showed that his multilinear utility functions observe Arrow's conditions. This result holds once more in our approach. Finally, these observations are extended to groups of statements.

We limit our search for the moment to single statements and therefore examine

$P[a\in \mathcal{A}_\varphi(A)]$, $P[a\in T_\varphi]$, $P[a\in F_\varphi]$

The respective quantities for groups of statements are examined later. The components are assumed to be NAND consistent and that they decide independently of each other. Then,

$P\ [a\in \mathcal{A}_\varphi]=P[\varphi(\mathbf{x}(a|A))=1]=h(\mathbf{p}(a|A))$

From reliability theory we know that $h(\mathbf{p})$ is linear in each p_i. Further,

$$\frac{\partial h(\mathbf{p})}{\partial p_i} \geq 0 \qquad \text{for all } i \tag{12.30}$$

because

$$h(\mathbf{p}) = E\varphi(\mathbf{x}) = E[x_i\varphi(1_i, \mathbf{x}) + (1-x_i)\varphi(0_i, \mathbf{x})]$$
$$= p_i E\varphi(1_i, \mathbf{x}) + (1-p_i)E\varphi(0_i, \mathbf{x})$$

taking partial derivatives w.r.t. p_i,

$$\frac{\partial}{\partial p_i}h(\mathbf{p}) = E[\varphi(1_i, \mathbf{x}) - \varphi(0_i, \mathbf{x})] \tag{12.31}$$

but $\varphi(1_i, \mathbf{x}) \geq \varphi(0_i, \mathbf{x})$ since φ is coherent and thus (12.30) is proved.

In particular,

$$\frac{\partial}{\partial p_i}h(\mathbf{p}(a)) > 0 \qquad \text{for all } i \tag{12.32}$$

whenever $0 < p_j(a) < 1$ for all $j \in \{1, \ldots, n\}$.

This holds because there is $\mathbf{x}^+$ so that $\varphi(1_i, \mathbf{x}^+) - \varphi(0_i, \mathbf{x}^+) = 1$ for some $\mathbf{x}^+$, otherwise component i is irrelevant, contrary to our definition of coherent structures, which must contain only relevant components. But now $\mathbf{x}^+(a)$ has positive probability of occurring because $0 < p_j(a) < 1$ for all j.

Arrow's utilities, as paraphrased for cardinal ordering, take the following form:

Axiom 1 on connectedness of preference ordering and **Axiom** 2 on transitivity of orderings are trivially satisfied once $h(\mathbf{p}(a|A))$ implies a cardinal ordering on alternatives

(statements).

Condition 1

(a) The number of elements (statements) in the set of alternatives A, is greater than or equal to three.

(b) $h(\mathbf{p})$ is defined for any $0<\mathbf{p}<1$

(c) There are at least three components.

Condition 2 (Positive responsiveness)

The function $h(\mathbf{p})$ is non decreasing in each p_i.

Condition 3 (Irrelevant alternatives)

If an alternative is added or subtracted from A, $h(\mathbf{p}(a|\ A))$ will remain unchanged.

Condition 4 (Preferences are not imposed)

For any statements a and b in A, there is $\mathbf{p}(a|A)$ and $\mathbf{p}(b|A)$ so that $h(\mathbf{p}(a|A))>h(\mathbf{p}(b|A))$.

Condition 5 (No dictator)

There is no component i so that whenever $p_i(a)\geq p_i(b)$ then $h(\mathbf{p}(a))\geq h(\mathbf{p}(b))$ regardless of the preferences $(p_j(a),\ p_j(b), j\in\{1,\ldots,n\}-\{i\})$ of the rest of the components.

Condition 1 is satisfied. Condition 2 is satisfied because of (12.32). Condition 4 is satisfied because of (12.32) and the fact that $h(\mathbf{1})=1$, $h(\mathbf{0})=0$. Therefore, by picking an arbitrarily large $\mathbf{p}(a)$ and an arbitrarily small $\mathbf{p}(b)$, we can make $h(\mathbf{p}(a))>h(\mathbf{p}(b))$.

Condition 5 is also satisfied since the existence of a dictator would imply that the rest of the components are irrelevant which contradicts our definition of coherent structures. Condition 3 will be satisfied as long as $p_i(a)$ for each component i is expressed regardless of the set *A*. If, however, it depends on *A* (i.e. $p_i(a|A)$), then Condition 3 will be violated at the component level and consequently will not hold at the structure level as well. By the same reasoning Condition 3 does not hold at the component and hence at the structure level, when components are restricted to accept a specific number of statements.

Let us examine now $P[a \in T_\varphi]$. Since components are NAND consistent

$$P[a \in T_\varphi] = P[\varphi^D(\mathbf{x}(a) = 1 \text{ and } \varphi(\mathbf{x}(a)) = 1]$$
$$= P[\varphi^D(\mathbf{x}(a))\varphi(\mathbf{x}(a)) = 1]$$

But $\varphi^D\varphi$ is a series structure consisting of φ^D in series with φ. The overall structure $\varphi^D\varphi$ is again a coherent structure, say ψ, which is self dual. Hence,

$$P[a \in T_\varphi] = P[\psi(\mathbf{x}(a)) = 1]$$
$$= h_\psi(\mathbf{p}(a))$$

Once more h_ψ satisfies the Arrow conditions and therefore so does $P[a \in T_\varphi]$. Similar observations hold for $P[a \in F_\varphi]$; but now it is strictly decreasing. To avoid this we may consider $1-P[a \in F_\varphi]$.

For groups of statements we need to examine the properties of, $P[(a_1,\ldots,a_q) \in \mathcal{A}_\varphi]$ or $P[(a_1,\ldots,a_q) \in \hat{T}_{\varphi q}]$ etc.

In particular,

$P[(a_1,\dots,a_q)\in\mathcal{A}_\varphi]=P[\Pi^q{}_{j=1}\varphi(\mathbf{x}(a_j))=1]$

and when the statements are independent,

$=\Pi^q{}_{j=1}P[\varphi(\mathbf{x}(a_j))=1]$

and the components are independent,

$=\Pi^q{}_{j=1}h(\mathbf{p}(a_j))$

Thus,

$$\frac{\partial}{\partial p_i(a_k)}\prod_{j=1}^{q} h(\mathbf{p}(a_j))=$$

$=\Pi^q{}_{j\neq k}h(\mathbf{p}(a_j))E[\varphi(1_i,\mathbf{x}(a_k))-\varphi(0_i,\mathbf{x}(a_k))]$

and when $0<p_i(a_j)<1$ for all i, j then,

$$\frac{\partial}{\partial p_i(a_k)}\prod_{j=1}^{q} h(\mathbf{p}(a_j))>0$$

and satisfaction of Arrow's conditions follows again.

Another quantity of interest at the structure level is,

$P[(a_1,\dots,a_q)\in\hat{T}_{\varphi q}]=$

when the components are NAND consistent,

$=P[\varphi^D(\Pi^q{}_{j=1}\mathbf{x}(a_j))=1$ and $\Pi^q{}_{j=1}\varphi(\mathbf{x}(a_j))=1]$

$=P[\varphi^D(\Pi^q{}_{j=1}\mathbf{x}(a_j))\Pi^q{}_{j=1}\varphi(\mathbf{x}(a_j))=1]$

But now $\Pi^q{}_{j=1}\varphi(\mathbf{x}(a_j))$ is equivalent to a series of q coherent structures $\varphi(\mathbf{x}(a_1))$, $\varphi(\mathbf{x}(a_2)),\dots,$ $\varphi(\mathbf{x}(a_q))$ which is equivalent to a coherent structure say $\psi_1(\mathbf{x}(a_1),\mathbf{x}(a_2),\dots,\mathbf{x}(a_q))$, while $\varphi^D(\Pi^q{}_{j=1}\mathbf{x}(a_j))$ is a coherent structure where each component i is replaced by a series of components $x_i(a_1)x_i(a_2)\dots x_i(a_q)$. It is then equivalent to a coherent structure $\psi_2(\mathbf{x}(a_1),\dots,\mathbf{x}(a_q))$. Then the overall

structure is the series structure $\psi_1\psi_2$, say $\psi(\mathbf{x}(a_1),\ldots,\mathbf{x}(a_q))$ and is coherent. Therefore we can write,

$P[(a_1,\ldots a_q)\hat{T}_{\varphi q}]=P[\psi(\mathbf{x}(a_1),\ldots,\mathbf{x}(a_q)=1]$

and when components and statements are independent

$$=h_\psi(\mathbf{p}(a_1),\ldots,\mathbf{p}(a_q))$$

Since ψ is coherent

$$\frac{\partial}{\partial p_i(a_k)}h_\psi(\mathbf{p}(a_1),\ldots,\mathbf{p}(a_q))>0$$

when $0<p_i(a_k)<1$ for all i,k. Thus again Arrow's conditions hold. Similar observations can be made for the probabilities that $(a_1,\ldots,a_q)$ belongs to $\hat{F}_{\varphi q}$ or to $\check{T}_{\varphi q}$ or to $\check{F}_{\varphi q}$.

BOOK 5

ABSTENTIONS AND HIERARCHIES

Chapter 13

ABSTENTIONS

13.1 Motivating Philosophy

In this study we assumed that each component in a coherent structure was able to decide whether to accept or reject an issue. The possibility of abstaining i.e. refusing to express his opinion, was not permitted. The same restriction was true when components were obliged to accept a specified number of statements from a given set.

However, real life presents us with a multitude of cases where components abstain. It is, therefore, the purpose of this section to study and incorporate the possibility of abstentions of components in the study of coherent decision structures.

First we must differentiate between the contradictory (accepting both $\bar{a}$ and a) and blocked (rejecting both a and $\bar{a}$) behaviour of components on some issues on the one hand, and abstention on the other. This case of contradictory or blocked behaviour does not arise if the components are NAND consistent. If, however, we allow the possibility for components to violate NAND consistency, then contradictory or blocked behaviour will possibly appear.

Contradictory or blocked behaviour is not equivalent to abstention, unless such an agreement exists among participating components. In the first case, contradictory or blocked votes of a component will be counted for or against the issues a and $\bar{a}$ according to the coherent structure φ, while in the second case of abstention the component is eliminated and a new structure φ', which does not contain the abstaining component(s), takes over instead of the original φ as we will see later.

The notion of a contradictory or blocked component is not merely of theoretical interest. There are plenty of examples in real life: Think of coherent structures whose components are themselves coherent structures. Think of unions which are represented in federations which are represented in confederations etc. It is quite possible that the coherent structure representing the decisions of some union will be blocked and thus the union thought of as a component within the federation will be blocked.

Returning now to abstentions we want to see what their effect is in changing the structure.

Starting from real life experience, we see that in k/n structures, when some components (say s in number) abstain, the structure changes to $k_s/n-s$ where k_s is appropriately chosen.

Take for example a group of people that decides according to majority rule ((n+1)/2 out of n, with n odd). If some people, say s, abstain, then they are completely disregarded and the rest, n-s, decide according to the majority rule, (n-s+1)/2 out of n-s; or approximately so if n-s is not odd. We are motivated, therefore, to make our definition of *abstention*.

Let N be the set of components of a coherent structure φ, and A a set of issues.

Definition

We will say that the set S ($S \subseteq N$) of components is abstaining from the structure φ for the decision on issue $a \in A$, if for the decision of that issue, a, the structure φ is replaced by a new structure φ_S which contains the

set of N-S components. The structure φ_S will be called the *abstention structure*.
Let also $|S|=s$.

The question still remains on how φ_S must be chosen. It is logical to require that in the new structure the components must keep the same relative strength, whatever it may be defined to be, as in the original structure, or at least as close as possible. In the next section we will define such a measure of relative power of components or groups of them.

13.2 The Expected Passing and Blocking Power of a Group of Components

Definition (Passing power of group G in structure φ)
The expected probability, $Y(\varphi, G)$, that a coherent structure φ, that contains the set $N=\{1,\ldots,n\}$ of components, will accept an issue when all components in group G ($G \subseteq N$) accept it, is called the *Passing Power of group G in structure φ* and is defined as,

$$Y(\varphi, G) \equiv E_{\mathbf{p}}[E_{\mathbf{x}|\mathbf{p}}\varphi(\mathbf{1}_G, \mathbf{x})] \qquad (13.1)$$

where

$\varphi(\mathbf{1}_G, \mathbf{x})$ is a fixing of $\varphi(\mathbf{x})$ with $x_i=1$ for $i \in G$

$$p_j \equiv P[x_j=1], \quad j \in N$$

$E_{\mathbf{x}|\mathbf{p}}\varphi(\mathbf{1}_G, \mathbf{x}) = P[\varphi(\mathbf{1}_G, \mathbf{x})=1 \mid \text{given the parameters } p_1,\ldots,p_n]$

and

$E_{\mathbf{p}}[E_{\mathbf{x}|\mathbf{p}}\varphi(\mathbf{1}_G, \mathbf{x})] \equiv \int_{(0,1)} E_{\mathbf{x}|\mathbf{p}}\varphi(\mathbf{1}_G, \mathbf{x})\, dF_1(p_1)\ldots dF_n(p_n)$

where the integral is from 0 to 1 and $F_i(p_i)$ is the cumulative probability distribution of p_i.

As usual, the notation $E_{\mathbf{p}}$ means that the expectation is taken with respect to the probability distribution of the random variable $\mathbf{p}$, while $E_{\mathbf{x}|\mathbf{p}}$ means that it is taken w.r.t. the random variable $\mathbf{x}$ conditioned on $\mathbf{p}$.

To avoid complicated notation, let

$$E_{\mathbf{x}|\mathbf{p}}\varphi(\mathbf{1}_G, \mathbf{x}) = h_\varphi(\mathbf{1}_G, p)$$

where we let $p_1=\ldots=p_n=p$

Similarly define,

Definition (Blocking power of group G in structure φ)

The expected probability $B(\varphi, G)$, that a coherent structure φ will block (reject) an issue when all components in group G ($G \subseteq N$) reject it is called the *Blocking Power of group G in structure φ* and is defined as,

$$B(\varphi, G) \equiv E_{\mathbf{p}}[P[\varphi(\mathbf{0}_G, \mathbf{x}) = 0 \mid \mathbf{p}]] \qquad (13.2)$$

Observe that

$$P[\varphi(\mathbf{0}_G, \mathbf{x}) = 0 \mid \mathbf{p}] = 1 - P[\varphi(\mathbf{0}_G, \mathbf{x}) = 1 \mid \mathbf{p}]$$
$$= 1 - E_{\mathbf{p}}[E_{\mathbf{x}\mid\mathbf{p}}\varphi(\mathbf{0}_G, \mathbf{x})]$$

If we let $p_1 = \ldots = p_n = p$ then,

$$B(\varphi, G) = 1 - E_p[h(\mathbf{0}_G, p)]$$
$$= E_p[h^D(\mathbf{1}_G, (1-p))]$$

and if p is uniformly distributed over (0, 1) then,

$$B(\varphi, G) = E_p[h^D(\mathbf{1}_G, p)$$
$$= Y(\varphi^D, G) \qquad (13.3)$$

Remark 1

In the following we will assume that $p_1 = \ldots = p_n = p$ and that p is uniformly distributed over (0, 1). Then, as we showed above, the blocking power of group G in φ is equal to the passing of G in φ^D, the dual structure.

Remark 2

Let $\varphi_G(x_G, \mathbf{x})$ denote the structure that results when we contract G in φ to a single component G, where x_G is the characteristic function of component G, who represents the original group G in the original structure φ. If φ has n components, then φ_G has n-g+1 components, where we let

$g=|G|$. Note that in previous chapters the symbol G was used as a shorthand for $|G|$. Now though, G refers to the component after contraction of group G and thus we use g when we refer to the cardinality of G.

Observe now that,

$$\varphi(\mathbf{1}_G, \mathbf{x}) = \varphi_G(1_G, \mathbf{x}) \text{ and } \varphi(\mathbf{0}_G, \mathbf{x}) = \varphi_G(0_G, \mathbf{x})$$

because according to the definition of contraction,

$$\varphi_G(x_G, \mathbf{x}) = x_G \varphi(\mathbf{1}_G, \mathbf{x}) + (1-x_G)\varphi(\mathbf{0}_G, \mathbf{x})$$

It follows that the passing or blocking power of group G in φ is the same as the passing or blocking power of component G in φ_G. Thus,

$$Y(\varphi, G) = Y(\varphi_G, G)$$

$$B(\varphi, G) = B(\varphi_G, G) \qquad (13.4)$$

Proposition 13.1

The passing power of component-i in structure φ that has n components is given by,

$$Y(\varphi, i) = \Sigma^n_{r=1} \psi(\varphi, r, i)(r-1)!(n-r)!/n! \qquad (13.5)$$

where $\psi(\varphi, r, i)$ is the number of paths (not only min paths) of φ of length r that contain component i.

Proof:

By definition

$$Y(\varphi, i) = \int_{(0,1)} h(1_i, p)\,dp \qquad (13.6)$$

But $h(p) \equiv E_{\mathbf{x}}\varphi(\mathbf{x})$ and also any coherent structure $\varphi(\mathbf{x})$ can be decomposed to

$$\varphi(\mathbf{x}) = \Sigma_{\mathbf{y}} \Pi^n_{j=1} x_j^{\,y(j)} (1-x_j)^{1-y(j)} \varphi(\mathbf{y})$$

as we recall from section 1.1. Since x_i's are identically distributed ($p_1 = \ldots = p_n = p$) and each y_j is a 0,1 variable.

$$h(p) = E_{\mathbf{x}}\varphi(\mathbf{x}) = \Sigma_{\mathbf{y}} p^{\Sigma y(j)} (1-p)^{n-\Sigma y(j)} \varphi(\mathbf{y}) \quad (13.7)$$

In our particular case, using (13.7), (13.6) becomes,

$$Y(\varphi, i) =$$
$$= \int_{(0,1)} [\Sigma_{\mathbf{y}} p^{\Sigma y(j)} (1-p)^{n-\Sigma y(j)} \varphi(1_i, \mathbf{y})] dp \quad (13.8)$$

where the summation Σy_j in the exponents is carried over all $j \neq i$. But the y_j's are 0, 1 variables and therefore (13.8) can be rearranged as follows,

$$Y(\varphi, i) = \Sigma^n_{r=1} \Psi(\varphi, r, i) \int_{(0,1)} p^{r-1} (1-p)^{n-r} dp$$
$$= \Sigma^n_{r=1} \Psi(\varphi, r, i) (r-1)!(n-r)!/n! \qquad //$$

We know that the passing power of a group G in a structure φ that contains n components in total, is equal to the passing power of component G in the contracted structure φ_G that contains n-g+1 components. Thus,

$$Y(\varphi, G) = Y(\varphi_G, G)$$
$$= \Sigma^n_{r=1} \Psi(\varphi_G, r, G) (r-1)!(n-g+1-r)!/(n-g+1)! \quad (13.9)$$

Let us now point the relation of $Y(\varphi, G)$ and $B(\varphi, G)$ (or $Y(\varphi^D, G)$) to the Shapley value. The Shapley value of a game u to player i, denoted as $I(u, i)$ is defined as,

$$I(u, i) =$$
$$= \Sigma_{S \subseteq N} [u(S) - u(S - \{i\})] (s-1)!(n-s)!/n! \quad (13.10)$$

where

N: the set of participating players

S: a subset of N

$s = |S|$, $n = |N|$

In case that u is a coherent structure φ, the reward $u(S)$

to group S is given by

$$u(S) = \begin{cases} 1 \text{ if } S \text{ is a path of } \varphi \\ \\ 0 \quad \text{otherwise} \end{cases}$$

Thus,

$$u(S) = \varphi(\mathbf{1}_S, \mathbf{0}_{N-S})$$

$$u(S-\{i\}) = \varphi(\mathbf{1}_{S-\{i\}}, \mathbf{0}_{(N-S)\cup\{i\}})$$

Then the term $u(S)-u(S-\{i\})$ of (13.10) is 1 whenever S is a path containing i and so that if i is excluded, S ceases to be a path (i.e. i is critical for S).

Thus (13.10) becomes,

$$I(\varphi, i) = \Sigma^{n}{}_{s=1}\eta(\varphi, s, i)(s-1)!(n-s)!/n! \qquad (13.11)$$

where,

$\eta(\varphi, s, i)$: The number of paths of length s (not only min paths) of φ, that contain i and which cease to be paths if i is excluded from them. (i.e. the number of paths of length s for which i is critical).

Observe the difference in the definition of $\eta(\varphi, s, i)$ and $\psi(\varphi, s, i)$. The latter does not require that i is critical in the counting of paths that contain i.

Using the notion of contraction we can talk about the Shapley value of a group of components instead of a single component i,

$$I(\varphi, G) =$$
$$= \Sigma^{n-g+1}{}_{s=1}\eta(\varphi_G, s, G)(s-1)!(n-g+1-s)!/(n-g+1)! \qquad (13.12)$$

Now we can show that what Barlow and Proschan [1974] define as *structural importance* of i in φ is the same as the Shapley value $I(\varphi, i)$ to player i (component i) of game φ.

Structural importance of i in φ is defined to be equal to

$\int_{(0,1)}[h(1_i,p)-h(0_i,p)]dp=$

but by the definition of $Y(\varphi,i)$ and $B(\varphi,i)$,

$$=Y(\varphi,i)+B(\varphi,i)-1 \qquad (13.13)$$

Still using the decomposition formula for $\varphi(\mathbf{x})$,

$$=\int_{(0,1)}\Sigma_{\mathbf{x}}[\varphi(1_i,p)-\varphi(0_i,p)]p^{\Sigma x(j)}(1-p)^{n-1-\Sigma x(j)}]dp$$

where Σx_j in the exponent is carried over all $j\neq i$. Thus,

$\int_{(0,1)}[h(1_i,p)-h(0_i,p)]dp=$

$$=\Sigma^n_{s=1}\eta(\varphi,s,i)\int_{(0,1)}p^{s-1}(1-p)^{n-s}dp$$

$$=\Sigma\eta(\varphi,s,i)(s-1)!(n-s)!/n!$$

$$=I(\varphi,i) \qquad (13.14)$$

as we wanted to show.

If instead of component i we talk about the group G then,

$$I(\varphi,G)=I(\varphi_G,G)=Y(\varphi_G,G)+B(\varphi_G,G)-1$$

Observe that

$$\Sigma^n_{i=1}I(\varphi,i)=1$$

because $h(1_i,p)-h(0_i,p)$ represents the probability that the system h operates (=1) because component i operates (and will fail if i fails) when all other components are operating with probability p each. Therefore,

$$\Sigma^n_{i=1}h(1_i,p)-h(0_i,p)=1$$

hence,

$$\Sigma^n_{i=1}I(\varphi,i)=\Sigma^n_{i=1}E_p[h(1_i,p)-h(0_i,p)]=1$$

This does not imply that the same property holds for $Y(\varphi,i)$ and $B(\varphi,i)$ separately, as the following example shows,

Example 13.2

Consider the two component series structure,

$\varphi(\mathbf{x})=x_1x_2$, $h(p)=p^2$, $h(1_1,p)=h(1_2,p)=p$, $h(0_1,p)=h(0_2,p)=0$

$Y(\varphi,1)=Y(\varphi,2)=\int_{(0,1)} p\,dp=1/2$

$B(\varphi,1)=B(\varphi,2)=1-0=1$

Therefore,

$\Sigma^2_{i=1} I(\varphi,i)=$

$$=\Sigma\, Y(\varphi,i)+\Sigma\, B(\varphi,i)-2=1$$

as expected.

In the example below the passing and blocking power of components or groups of them is calculated when φ is a symmetric k/n structure.

Example 13.3

When φ is a symmetric structure k/n, then

$$Y(\varphi,G)=(n-k+1)/(n-g+1) \tag{13.16}$$

where $g=|G|\leq k-1$, otherwise $Y(\varphi,G)=1$. This is true because $Y(\varphi,G)=Y(\varphi_G,G)$

But φ_G consists of those paths of φ that either do not contain any component of G or contain all of G as we recall from our investigation on contractions in Chapter 5. Therefore,

$$\psi(\varphi_G,r,G)=\begin{cases} 0 & \text{if } r\leq k-1 \\ \binom{n-g}{r-1} & \text{if } r\geq k \end{cases}$$

Then using (13.9)

$Y(\varphi,G)=Y(\varphi_G,G)=$

$$=\Sigma^n_{r=k}\ 1/(n-g+1)$$

$$=(n-k+1)/(n-g+1)$$

Also

$$B(\varphi, G) = Y(\varphi^D, G) = K/(n-g+1)$$

(13.17)

The structural importance of group G in φ is,

$I(\varphi, G) =$

$$= Y(\varphi, G) + Y(\varphi^D, G) - 1$$

$$= g/(n-g+1)$$

and when $g=1$, $G=\{i\}$, then $I=1/n$ as expected.

13.3 Conditions for Fair Abstentions

If appropriate restrictions are not placed, it is possible that a group of cooperating components may decide that some or all of their members must abstain in the hope that when the new structure takes over, they will have higher probability of passing or blocking issues.
The following conditions are proposed to safeguard against that possibility.
Let $G_1 \cup S$ $(G_1 \cap S = \emptyset)$ be a group of cooperating components in structure φ, and let φ_S be the structure taking over when S abstains.

Condition 1

Abstentions must lead to non increasing passing and blocking power:

$$Y(\varphi, G_1 \cup S) \geq Y(\varphi_S, G_1) \qquad (13.18)$$

and

$$Y(\varphi^D, G_1 \cup S) \geq Y(\varphi^D{}_S, G_1) \qquad (13.19)$$

This condition includes the requirement that if a group $G_1 \cup S$ was not a path in φ then G_1 will not be a path in φ_S. And also if $G_1 \cup S$ is not a cut in φ, then G_1 will not be a cut in φ_S.

Condition 2

The group $G_1 \cup S$ choosing abstention for a subset S of

its components, must loose both passing and blocking power relative to any other group G_2 that decides not to abstain:
For any group S of components that abstain and any pair of groups of components G_1, G_2,

$$\frac{Y(\varphi_S, G_1)}{Y(\varphi, G_1 \cup S)} \leq \frac{Y(\varphi_S, G_2)}{Y(\varphi, G_2)} \tag{13.20}$$

and

$$\frac{Y(\varphi^D{}_S, G_1)}{Y(\varphi^D, G_1 \cup S)} \leq \frac{Y(\varphi^D{}_S, G_2)}{Y(\varphi^D, G_2)} \tag{13.21}$$

Condition 3

The ratio of passing over blocking power before and after abstention must remain unchanged for all groups:
For any group S of components that abstain and any group G_i so that $G_i \cap S = \emptyset$, both (a) and (b) hold:

(a)

$$\frac{Y(\varphi, G_i \cup S)}{Y(\varphi^D, G_i \cup S)} = \frac{Y(\varphi_S, G_i)}{Y(\varphi_S{}^D, G_i)} \tag{13.22}$$

(b)

$$\frac{Y(\varphi, G_i)}{Y(\varphi^D, G_i)} = \frac{Y(\varphi_S, G_i)}{Y(\varphi^D{}_S, G_i)} \tag{13.24}$$

In words Condition 3 says that if $G_i \cup S$ was an original group (before abstention) of cooperating components that decides to abstain by S then (a) must hold. If G_i is an original

group that does not abstain then (b) must hold.

Observe that Conditions 2 and 3 imply that,

$$\frac{Y(\varphi_S, G_1) + Y(\varphi^D{}_S, G_1)}{Y(\varphi, G_1 \cup S) + Y(\varphi^D, G_1 \cup S)} \leq$$

$$\leq \frac{Y(\varphi_S, G_2) + Y(\varphi^D{}_S, G_2)}{Y(\varphi, G_2) + Y(\varphi^D, G_2)} \tag{13.25}$$

which says that the group $G_1 \cup S$ that decides to partially abstain with S will loose total passing and blocking power at a greater proportion than any other group G_2 that does not abstain. To show the validity of (13.25) start with Condition 3,

$$\frac{Y(\varphi^D{}_S, G_1)}{Y(\varphi^D, G_1 \cup S)} = \frac{Y(\varphi_S, G_1)}{Y(\varphi, G_1 \cup S)}$$

and therefore,

$$= \frac{Y(\varphi^D{}_S, G_1) + Y(\varphi, G_2)}{Y(\varphi^D, G_1 \cup S) + Y(\varphi, G_1 \cup S)}$$

Also for G_2 that does not abstain,

$$\frac{Y(\varphi^D{}_S, G_2)}{Y(\varphi^D, G_2)} = \frac{Y(\varphi_S, G_2)}{Y(\varphi, G_2)}$$

and therefore,

$$= \frac{Y(\varphi^D{}_S, G_2) + Y(\varphi_S, G_2)}{Y(\varphi^D, G_2) + Y(\varphi, G_2)}$$

Using now Condition 2, (13.20), the result is obtained.

Definition (Fair Abstention Structure)

The structure φ_S resulting from the abstention of a group of components S of the original structure φ, will be called a *fair abstention structure* if Conditions 1, 2, 3 are satisfied.

13.4 Fair and Almost Fair Abstention on Symmetric Structures

The aim of this section is to see what form Conditions 1, 2, 3 take when the structure φ is a symmetric structure (k/n). As usual $G_1 \cup S$ is a group of components that decides to partially abstain with its subset S. The abstention structure is denoted φ_s $(k_s/n-s)$ where $s=|S|$. Further, G_2 is another group of components that does not decide to abstain.

In Example 13.3 it was shown that for φ symmetric (k/n) we have

$$Y(\varphi, G) = (n-k+1)/(n-g+1)$$

$$Y(\varphi^D, G) = k/(n-g+1)$$

and

$$I(\varphi, G) = Y(\varphi, G) + Y(\varphi^D, G) - 1$$

$$= g/(n-g+1)$$

Condition 1 requires that

$$Y(\varphi, G_1 \cup S) \geq Y(\varphi_s, G_1)$$

This implies that

$$(n-k+1)/(n-s-g_1+1) \geq (n-s-k_s+1)/(n-s-g_1+1)$$

which is equivalent to,

$$k_s \geq k-s \tag{13.26}$$

where $g_1 = |G_1|$

Also the requirement that

$$Y(\varphi^D, G_1 \cup S) \geq Y(\varphi^D{}_s, G_1)$$

implies

$$k/(n-g_1-s+1) \geq k_s/(n-s-g_1+1)$$

which is equivalent to,

$$k \geq k_s \tag{13.27}$$

From (13.26) and (13.27) we obtain

$$k-s \le k_s \le k \qquad (13.28)$$

which is necessary and sufficient for k/n structures to satisfy Condition 1.

Condition 2 requires that,

$$\frac{Y(\varphi_s, G_1)}{Y(\varphi, G_1 \cup S)} \le \frac{Y(\varphi_s, G_2)}{Y(\varphi, G_2)}$$

and using (13.16) and (13.17) the condition reduces to

$$s \ge 0 \qquad (13.29)$$

which is necessary and sufficient for Condition 2 to hold when the structure is symmetric. It is obvious that this condition is always satisfied since $|S| \ge 0$.

Condition 3 requires that,

$$\frac{Y(\varphi, G_1 \cup S)}{Y(\varphi^D, G_1 \cup S)} = \frac{Y(\varphi_s, G_1)}{Y(\varphi^D{}_s, G_1)}$$

and substituting the values of Y's from (13.16), (13.17) and after some manipulation we obtain,

$$k_s = (n-s+1)k/(n+1) \qquad (13.30)$$

Condition 3 also requires that,

$$\frac{Y(\varphi, G_2)}{Y(\varphi^D, G_2)} = \frac{Y(\varphi_s, G_2)}{Y(\varphi^D{}_s, G_2)}$$

and using (13.16) and (13.17) we obtain (13.30) again. Therefore, (13.30) is necessary and sufficient for Condition 3 to hold.

Since k_s must be integer, (13.30) will not be satisfied in general. If we define k_s' as,

$$k_s' = (n-s+1)k/(n+1) \tag{13.31}$$

then k_s can be chosen as the closest integer to k_s':

$$\text{either } k_s = [k_s'] \tag{13.32}$$

$$\text{or } k_s = [[k_s']] \tag{13.33}$$

as long as $k_s \leq n-s$

This observation leads to a new definition,

Definition

We will say that Condition 3 is *almost satisfied* by a symmetric abstention structure $k_s/(n-s)$, when k_s is chosen to be equal either to $[k_s']$ or to $[[k_s']]$ where k_s' is given by (13.31).

Lemma 13.4

Both $[k_s']$ and $[[k_s']]$ lie in the interval $[k-s,k]$.

Proof:

Let $s \geq 1$ (if $s=0$ then $k_s=k$ trivially)

(a)

$$\begin{aligned} [k_s'] &= [(n+1-s)k/(n+1)] \\ &= [k-sk/(n+1)] \\ &= k-[[sk/(n+1)]] < k \end{aligned}$$

Also since $k/(n+1) \leq 1$ then $sk/(n+1) \leq s$. Therefore,

$$[k-sk/(n+1)] \geq k-s \tag{13.34}$$

Hence,

$$k-s \leq [(n+1-s)k/(n+1)] \leq k$$

(b) Let first k_s' not be integer otherwise we are in case (a) above. Now,

$[[k_s']] = [k_s'] + 1 \geq k-s$ because of (13.34). Also,

$$[k - sk/(n+1)] + 1 \leq k \qquad (13.35)$$

because

$$k - sk/(n+1) < k \text{ thus, } [k - sk/(n+1)] \leq k-1$$

from which (13.35) follows by adding 1 to both sides. //

From Lemma 13.4 it follows that when k_s is chosen to be equal to $[k_s']$ or $[[k_s']]$, Condition 1 is satisfied. Further, Condition 3 is almost satisfied while Condition 2 is trivially satisfied as long as $s \geq 0$.

Remark

No matter how k_s is chosen, it must always satisfy $k_s \leq n-s$. Let us check if $[k_s']$ and $[[k_s']]$ satisfy this condition.

(a) If $k_s = [k_s']$

$k_s = [(n+1-s)k/(n+1)]$ and $k<n+1$ imply that $(n+1-s)k/(n+1) < n+1-s$ and thus,

$$[(n+1-s)k/(n+1)] \leq n-s$$

(b) If $k_s = [[k_s']]$

In this case if k is very close to n it is possible that

$$[(n+1-s)k/(n+1)] + 1 = n+1-s > n-s$$

If this situation occurs, we cannot choose $k_s = [[k_s']]$, but we have to restrict to $k_s = [k_s'] = n-s$, which is a series structure.

Because of our new definition of *almost satisfying* Condition 3, we need a new definition of *almost fair abstention*

Definition (Almost Fair Abstention)

Given a k/n structure, we will say that the symmetric structure

$k_s/n-s$ is an *almost fair abstention structure* if Conditions 1 and 2 hold and Condition 3 is almost satisfied.
We can summarize, therefore, with the following

Proposition 13.5

If k_s is chosen so that it is equal to $[k_s']$ or $[[k_s]]$ and as long as $k_s \leq n-s$, the structure $k_s/n-s$ is an almost fair abstention structure.

Proof:

Straightforward from Lemma 13.4. //

13.5 Almost Logic Invariant Abstentions on Symmetric Structures

It is now time to turn our attention to logic requirements. Namely, it is reasonable to ask that the following condition is satisfied when abstention occurs.

Condition L (Logic Invariance)
If $\varphi\in\mathcal{S}_p$ and $\varphi\notin\mathcal{S}_{p+1}$ then $\varphi_S\in\mathcal{S}_p$ and $\varphi_S\notin\mathcal{S}_{p+1}$ (and similarly for $\overline{\mathcal{S}}_p, \overline{\mathcal{S}}_{p+1}$)

In particular, for k/n structures, $\varphi\in\mathcal{S}_p$ is equivalent to

$$kp\geq n(p-1)+1 \tag{13.36}$$

while $\varphi\notin\mathcal{S}_{p+1}$ is equivalent to

$$k(p+1)<np+1 \tag{13.37}$$

From (13.36) and (13.37) we obtain

$$(k-1)/(n-k)<p\leq(n-1)/(n-k) \tag{13.38}$$

Denote

$$p_h=(n-1)/(n-k) \text{ and } p_l=(k-1)/(n-k)$$

Then $p_h-p_l=1$

It follows that there is only one integer in the interval $(p_l, p_h]$ where p belongs. Now for the abstention structure φ_s, which is $k_s/n-s$, we require similarly that $p_s\in(p_{sl}, p_{sh}]$ where

$$p_{sl}=(k_s-1)/(n-s-k_s) \text{ and } p_{sh}=(n-s-1)/(n-s-k_s) \tag{13.39}$$

If we demand that $p_{sh}=p_h$ then it will follow that $p_l=p_{sl}$ also, and thus $p=p_s$, as there is only one integer in the interval $(p_l, p_h]$. As a consequence, Condition L will be satisfied. However, if we set $p_{sh}=p_h$ and solve for k_s we get

$$k_s = \frac{s+k(n-s-1)}{n-1}$$

which, in general, is not integer. We are led, therefore, in defining

$$k_s'' = \frac{s+k(n-s-1)}{n-1} \quad (13.40)$$

and then letting

$$\text{either } k_s = [k_s''] \quad (13.41)$$

$$\text{or } k_s = [[k_s'']] \quad (13.42)$$

as long as $k_s \leq n-s$

Once k is defined by (13.41) or (13.42) then, $p_h(k) \neq p_{sh}(k_s)$ and therefore Condition L will not be satisfied; but as $p_{sh}(k_s)$ is now the closest possible to $p_h(k)$ (from above or below) so that k_s is an integer, we know that Condition L will be the closest possible to be satisfied. A new definition is therefore needed.

Definition

Given a k/n structure and its abstention structure $k_s/n-s$, we will say that Condition L is *almost satisfied* if k_s is chosen to be equal either to $[k_s'']$ or $[[k_s'']]$.

It is easy to show that whether k_s is chosen according to (13.41) or (13.42), Condition 1 will still be satisfied

namely, both

$$k-s \leq [\frac{s+k(n-s-1)}{n-1}]$$

and

$$\left[\frac{s+k(n-s-1)}{n-1}\right]+1\leq k$$

hold. To show this it suffices to prove that,

(a) $k-s\leq(s+k(n-s-1))/(n-1)$, because $k-s$ is integer, and

(b) $\frac{s+k(n-s-1)}{n-1}+1\leq k$ because k is integer

Both (a) and (b) can be shown to hold by algebraic manipulations. In particular for (b) we obtain $s+n-1\leq sk \Rightarrow s\geq(k-1)/(n-1)$ which holds because $s\geq1$ by assumption.

Condition 2 is trivially satisfied as long as $s\geq0$. It only remains to check the interrelation between Condition 3 and Condition L for k/n structures.

For Condition 3 to be almost satisfied, k_s must be chosen equal to $[k_s']$ or $[[k_s']]$ while for Condition L to be almost satisfied k_s must be equal either to $[k_s'']$ or to $[[k_s'']]$. Let's try to unify those conditions.

First observe that

$$k_s'-k_s''=\frac{n-s+1}{n+1}k-\frac{s+k(n-s-1)}{n-1}$$

$$=s(2k-n-1)/(n+1)(n-1)\quad\begin{cases}\geq0 \text{ if } k\geq(n+1)/2\\ \\ \leq0 \text{ if } k\leq(n+1)/2\end{cases}$$

Let us concentrate for the moment to the case $k\geq(n+1)/2$. The case

when $k \leq (n+1)/2$ will follow by symmetrical arguments.

Now since $k \leq n$,

$s(2k-n-1)/(n+1)(n-1) \leq s(2n-n-1)/(n+1)(n-1) = s/(n+1) < 1$, since $s \leq n$.

Therefore,

$$0 \leq k_s' - k_s'' < 1 \qquad (13.43)$$

which means that between k_s' and k_s'' there is at most one integer and that $k_s'' \leq k_s'$ for $k \geq (n+1)/2$.

There are three cases to be examined:

Case 1: There is no integer between k_s'' and k_s' and neither is integer. Then pictorially we have,

k_s'' k_s'

$[k_s'']$ $[[k_s'']] = [k_s']$ $[[k_s']]$

It follows that if we choose

$$k_s = [[k_s'']] = [k_s'] \qquad (13.44)$$

Conditions 1,2 will be satisfied and Conditions 3 and L will be almost satisfied.

Case 2: There is no integer between k_s'', k_s' and neither is integer. Then pictorially we have,

k_s'' k_s'

$[k_s''] = [k_s']$ $[[k_s'']] = [[k_s']]$

Then if we let $k_s=[k_s'']=[k_s']$ or $k_s=[[k_s'']]=[[k_s']]$ when $[[k_s']]\leq n-s$ is satisfied, Conditions 1, 2 are satisfied and Condition 3 and L are almost satisfied.

Case 3: Either k_s'' or k_s' is integer.
Since $0\leq k_s''-k_s'<1$ it follows that if we choose k_s to be equal to whichever of k_s'' or k_s' is integer, once again Conditions 1, 2 are satisfied and Conditions 3 and L are almost satisfied (in fact one of them, 3 or L, is exactly satisfied).

Remark 1

When $k\leq(n+1)/2$ we know that the k/n structure belongs to $\overline{S}_2$. The same results hold as when $k\geq(n+1)/2$, but now min paths are replaced by min cuts. In particular, we can summarize the correspondence in the following table:

$k\geq(n+1)/2$	$k\leq(n+1)/2$
n	n
k	$n-k+1$
k_s	$n-s-k_s+1$
Condition 1	
$k_s\geq k-s$	$k_s\leq k$
$k_s\leq k$	$k_s\geq k-s$
Condition 2	
$s\geq 0$	$s\geq 0$
Condition 3	

Condition L

$(k-1)/(n-k)<p\leq(n-1)/(n-k)$	$(n-k)/(k-1)<p\leq(n-1)/(k-1)$

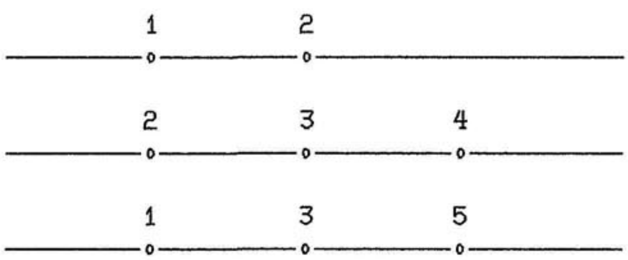
$$k_s''=\frac{(n-s-1)k+s}{n-1}$$

$k_s''\leq k_s'$	$k_s''\geq k_s'$
$0\leq k_s'-k_s''<1$	$0\leq k_s''-k_s'<1$

Cases 1, 2, 3 for picking $[k_s']$ or $[[k_s']]$ etc. can be easily retraced when $k\leq(n+1)/2$.

Example 13.6

Consider the coherent structure $\varphi(\mathbf{x})$ given by the following picture of min paths,



Then,

$\varphi(\mathbf{x})=x_1x_2+x_2x_3x_4+x_1x_3x_5-x_1x_3x_4-x_1x_2x_3x_5-x_1x_2x_3x_4x_5+x_1x_3x_4x_5$

$h(p)=E\varphi(\mathbf{x})=p^2+p^3-p^5$

$h(1_1,p)=E\varphi(1_1,\mathbf{x})=$

$=E(x_2+x_2x_3x_4+x_3x_5-x_3x_4-x_2x_3x_5-x_2x_3x_4x_5+x_3x_4x_5)$

$=p+p^3-p^4$

The passing power of component 1 is,

$Y(\varphi, 1) = \int_{(0,1)} h(1_1, p)\,dp = \int (p+p^3-p^4)\,dp = 11/20$

$h(0_1, p) E\varphi(0_1, p) = E(x_2 x_3 x_4) = p^3$

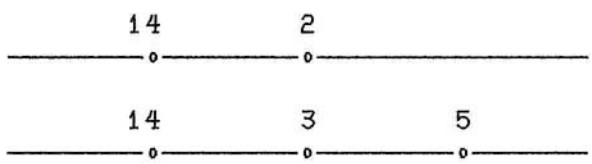
The blocking power of component 1 is

$B(\varphi, 1) = 1 - \int h(0_1, p)\,dp = 1 - \int p^3 dp = 3/4$

The total power or relative importance of component 1 is,

$I(\varphi, 1) = Y(\varphi, 1) + B(\varphi, 1) - 1 = 11/20 + 3/4 - 1 = 6/20$

Suppose now that components 1 and 4 are cooperating. Then φ contracts to φ_{14} and components 1, 4 collapse to a single component 14, while the min paths of φ_{14} are now:



$\varphi_{14}(\mathbf{x}) = x_{14}x_2 + x_{14}x_3x_5 - x_{14}x_2x_3x_5$

$Y(\varphi_{14}, 14) = Y(\varphi, \{1, 4\}) = \int (p+p^2-p^3)\,dp = 7/12$

$B(\varphi, \{1, 4\}) = B(\varphi_{14}, 14) = 1 - 0 = 1$ since component 14 is a cut by itself.

Example 13.7

Let a 61/100 symmetric structure φ. Suppose that in a voting situation 20 components abstain. We wish to determine the almost fair abstention structure φ_s which is of the form $k_s/n-s$.

$k_s' = (n-s+1)k/(n+1) = 48.92$

$k_s'' = ((n-s-1)k+s)/(n-1) = 48.87$

where $n=100$, $s=20$, $k=61$

Therefore, choosing

either $k_s = [[k_s']] = [[k_s'']] = 49$

or $k_s = [k_s'] = [k_s''] = 48$

while $n-s=80$, Conditions 3 and L are almost satisfied. But how

far from completely satisfying the conditions are we ?

(a) To check "how much" Condition 3 is violated, we calculate the ratio $R_3=A/B$ where $A=Y(\varphi_S, G_1)/Y(\varphi^D{}_S, G_1)$ and $B=Y(\varphi, G_1 \cup S)/Y(\varphi^D, G_1 \cup S)$.

R_3 should equal to 1 if Condition 3 is to be satisfied. Now in our case

$$R_3=k(n-s-k_s+1)/k_s(n-k+1)$$
$$=0.9959 \quad \text{for } k_s=49$$
$$=1.048 \text{ for } k_s=48$$

Indeed we are very close to satisfying Condition 3, especially when $k_s=49$.

(b) To evaluate how far from satisfying Condition L we are, we may calculate the difference

$$D=p_h-p_{sh}$$

which should equal to zero if Condition L were to be satisfied. Now

$$D=(n-1)/(n-k) \; - \; (n-s-1)/(n-s-k_s)$$
$$=0.01 \text{ for } k_s=49$$
$$=0.07 \text{ for } k_s=48$$

Luckily enough the choice of φ_S to be 49/80 is very close to satisfying both Conditions 3 and L.

Example 13.8

This example is based on the same ideas as those for abstention only the situation is different.

An international committee decides according to a k/n structure, call it φ, where each member is one component. New member countries (s in number) are going to join the committee and as a consequence the decision structure must be redefined to $k_s/n+s$

(φ_s) so that,

(a) The ratio of passing to blocking power of each component is preserved as close as possible.

(b) It can support the same level of logic complexity. Namely, if $\varphi \in \mathcal{S}_p$ and $\varphi \notin \mathcal{S}_{p+1}$ then $\varphi_s \in \mathcal{S}_p$ and $\varphi \notin \mathcal{S}_{p+1}$.

To guarantee (a) we require,

$$Y(\varphi, 1)/B(\varphi, 1) = Y(\varphi_s, 1)/B(\varphi_s, 1)$$

or

$$(n-k+1)/k = (n+s-k_s+1)/k_s'$$

Note that k_s' and k_s'' are given by the same formulas as in (13.31) and (13.40) only s is replaced by -s since now the structure φ is augmented by s components instead of being diminished as in the case of abstentions. The relation between k_s' and k_s'' is similar to the abstention case. Namely, $0 \leq k_s'' - k_s' < 1$ for $k \geq (n+1)/2$ (recall (13.43)) and the opposite for $k \leq (n+1)/2$. Therefore, (a) and (b) can be simultaneously satisfied by choice of k_s.

Chapter 14

HIERARCHIES

14.1 Definitions: Voting Situation, Hierarchy, Coherence.

Coherent structures and their properties were examined in previous chapters. The problem of inconsistent answers or no answers was characteristic. To overcome this difficulty in real life people use hierarhies. Consider for example an election for president in an association that takes place in two phases. In the first phase, if anyone of the candidates obtains more than half of the votes is elected. If no one does, then eliminate all candidates except those two that got most of the votes in the first phase and perform the second phase voting.

The idea of using hierarchies roughly means that a structure φ_1 is used first. if it fails to give answers or gives inconsistent answers, then another structure φ_2 takes over. If we are still not satisfied another structure φ_3 is activated etc. until some terminal structure resolves the problem. The terminal structure might as well be the one component structure (dictator) or a chance system or "status quo" (constant structure).

But hierarchies, as we will see, can be more broadly defined as algorithms for determining the overall desires of society (=a group of components).
To make the notion of hierarchy somehow more exact, we will first define the *voting situation*.

Definition

A *voting situation* (v. s.) is defined as the quadruplet $[A, \varphi, \mathbf{x}, F]$ where,

A: the set of statements

φ: a coherent structure

$\mathbf{x}$: the vector whose elements are $x_1, \ldots, x_n$

x_i: the characteristic function of component i

F: a restriction imposed on the choices of components. For example, $|\mathcal{A}_i| \geq \alpha$ or $|\mathcal{A}_i| \leq \alpha$ or $|\mathcal{A}_i| = \alpha$ or $|\overline{\mathcal{A}}_i| \geq \overline{\alpha}$ or $|\mathcal{A}_i| \leq \overline{\alpha}$ or $|\mathcal{A}_i| = \overline{\alpha}$.

Remark 1

Observe that $|\mathcal{A}_i| \geq \alpha$ is not equivalent to $|\overline{\mathcal{A}}_i| \leq A-\alpha = \overline{\alpha}$ neither $|\mathcal{A}_i| = \alpha$ to $|\overline{\mathcal{A}}_i| = \overline{\alpha}$ because abstentions are generally allowed.

Remark 2

We will distinguish among v. s.'s that allow abstentions and those that do not. If abstentions are allowed the method for obtaining the outcome of the v. s. must be clearly defined. Recall Chapter 13 on abstentions.

Remark 3

In the definition of the v. s. we include the possibility that φ is one of the following:

(a) The constant structure (status quo). Of course, this assumes that there is an alternative in A that represents the "status quo".

(b) The one component structure (dictator).

(c) The chance structure, where an alternative is randomly chosen from A according to some probability law.

Definition

A hierarchy of structures is a logic flow chart (algorithm) consisting of nodes that are (a) Voting Situations (b) Branching criteria (c) Stop decisions, and where the final outcome set(s) is (are) appropriately defined.

In general each Voting Situation requires a new voting. For example when A or α or $\mathbf{x}$ changes. But this is not always necessary. For example when $A, \alpha, \mathbf{x}$ remain the same but φ changes. Such a case appears for example when instead of 2/3 majority we change to 3/4 majority.

If a new voting takes place, the participating components may have total, partial or no information on the v.s.'s and their outcomes that took place before.

Hierarchies, as defined, can cover a wide variety of situations in real life. Take for example structures that decide on the issues to be presented for final decision to another structure, or structures that decide on the type of structure to be in force for a set of statements in A, or the group of components to participate in another v.s. that appears lower in the hierarchy, or the branching criteria to be used etc.

In this study we will limit ourselves to examining some simple and useful hierarchies, the *Simple Hierarchies*

Definition

Simple Hierarchy is called a hierarchy which when presented with a set of alternatives A, it constructs a sequence of sets, B_1, B_2,... so that

$\cup_j B_j = A$ and $B_i \cap B_j = \emptyset$ for all i, j

(As usual A is assumed finite and also the sequence of B_j's).

From the sequence of the sets B_j we can construct:

(a) A dichotomous decision (accept-reject) by assuming all elements within each B_j equivalent and applying the rule:

- Accept all statements in B_j for each $j \leq \mu$
- Reject all statements in B_j for each $j > \mu$

Examples of such dichotomous decisions are those that appear below in th *Ordering Algorithm* and the *Logic Hierarchy*.

(b) A *noncardinal ordering*, by assuming all statements within each B_j equivalent and applying the rule:

"B_j is preferred to B_i" if $j<i$

This can be applied for example to the *Ordering Algorithm* below.

(c) A *cardinal ordering* by defining a correspondence between B_j's and a sequence of numbers. Such a situation appears in th *Simple Electoral Ordering* below.

In the study of Simple Hierarchies two issues are important: The first is that of *completeness*. The algorithm is complete

if it terminates in a finite number of steps.

The second issue is that of *coherence*. Roughly coherence requires that the algorithm must respond positively to the desires of components. More precisely: Let $a \in A$ and $a \in B_l$ while for some component i in some Voting Situation $x_i(a)=0$. If in the same Voting Situation component i changes to $x_i'(a)=1$ while all other components in this and all other v.s.'s in the hierarchy do not change their opinion for any alternative in A (i.e. $x_j(c)=x'_j(c)$ for all $j \in \{1,...,n\}-\{i\}$) then $a \in B_{l'}$, with $l' \leq l$.

The requirement that hierarchies must be coherent is intuitively attractive because otherwise the hierarchy will be malignant by responding negatively, or in general not monotonically, to components' wishes.

Let us not confuse the idea of strategic voting with that of coherence of the algorithm. It is obvious that a component or a group of components will act so as to maximize the probability (or some other criterion) of having their individual choices be adopted by the hierarchy as a whole. Some strategy, therefore, will always be followed, depending, among other things, on the particular algorithm, and the information one has or expects to collect on how other components vote.

Let us also not confuse the concept of strategy with that of fairness. The latter has to do with giving equal opportunities and information to all components so that they may be able to apply whatever strategy they think best for their goal.

In the following sections some important examples of Simple Hierarchies will be studied.

14.2 Ordering Algorithm

The Ordering Algorithm orders a given set of alternatives A in equivalence classes using a hierarchy of symmetric k/n structures as k takes values from 1 to n.

The outcome sets $\mathcal{A}_{k/n}(A)$ of Voting Situations $[A, k/n, \mathbf{x}, \mathcal{A}_i \geq \alpha]$ for $k=1,\ldots,n$, imply an order on the alternatives in A, because $\mathcal{A}_{k/n} \supseteq \mathcal{A}_{k+1/n}$ while $A \supseteq \mathcal{A}_{1/n}$. It is clear that alternatives in $\mathcal{A}_{k+1}/n$ are preferred by the society of n people to alternatives in $\mathcal{A}_{k/n} - \mathcal{A}_{k+1/n}$ for all k. The elements of $\mathcal{A}_{n/n}$, whenever it is not empty, are the most wanted since they are accepted unanimously, while the opposite holds for $\mathcal{A}_{1/n}$. In short, the statements are ordered according to the number of votes they get.

In order to connect with the definition of simple hierarchies:

B_n: the set of statements that get n votes= $\mathcal{A}_{n/n}$

$B_{n-1} = \mathcal{A}_{n-1/n} - \mathcal{A}_{n/n}$

. .

. .

$B_1 = A_{1/n} - \mathcal{A}_{2/n}$

$B_0 = A - \mathcal{A}_{1/n}$

The number of statements $|\mathcal{A}_i|$ each component i accepts, will obey the restriction $|\mathcal{A}_i| \geq \alpha$ (or $|\mathcal{A}_i| = \alpha$ or $|\mathcal{A}_i| \leq \alpha$). If $|\mathcal{A}_i| \geq 0$ then component i is entirely free on the number of statements he may accept. It follows, that before we proceed with the Ordering Algorithm, we must examine whether there are α-plets of statements in A that are not identically false (i-false) under conjunction. If the ordering is

on rejections of statements, the relevant outcome sets are given by the sequence $\bar{\mathcal{A}}_{k/n}(A)$ $k=1,\ldots,n$ while the restrictions on the choice set of component i becomes $|\bar{\mathcal{A}}_i|\geq\bar{\alpha}$ or $|\bar{\mathcal{A}}_i|\leq\bar{\alpha}$. In that case we have to check if there are $\bar{\alpha}$-plets that are not identically true under disjunction. If no such α-plets ($\bar{\alpha}$-plets) exist, then components cannot pick an α-plet without violating their NAND consistency (NOR consistency) and therefore A or α must be modified by eliminating statements from A or diminishing α.

It is often required that one chooses a set $B\subseteq A$ so that B contains the β most wanted statements of A and so that $|B|\geq\beta$ (or $|B|\leq\beta$) while $||B|-\beta|$ is minimum. This is the case of a dichotomous decision (recall the definition of Simple Hierarchy). To achieve this we search for that k for which $|\mathcal{A}_{k/n}|=\beta$ and let $B=\mathcal{A}_{k/n}$. If this is not possible, we find instead k for which $|\mathcal{A}_{k/n}|\geq\beta$ and $|\mathcal{A}_{k+1/n}|<\beta$. Then let $B=\mathcal{A}_{k/n}$. However, if there is no k for which $|\mathcal{A}_{k/n}|\geq\beta$ we distinguish two cases: Either $\mathcal{A}_{1/n}\not\subseteq\emptyset$ in which case $|\mathcal{A}_{1/n}|<\beta$ and thus the best we can get is letting $B=\mathcal{A}_{1/n}$ but still we cannot satisfy $|B|\geq\beta$, or $\mathcal{A}_{1/n}=\emptyset$ which means that all components abstain on all statements.

The possibility of abstentions must be incorporated into the Ordering Algorithm. The simplest form of abstention is when components abstain totally. In this case either they accept statements in $\mathcal{A}_i$ and reject those in $A-\mathcal{A}_i$ or abstain totally from voting. When abstentions are of this simple form, the

sequence of structures $\{k/n\}$ for $k=1,\ldots,n$ in the Ordering Algorithm, will be replaced by the sequence $\{k_s/n-s\}$ for $k_s=1,\ldots,n-s$ in the sequence of Voting Situations, where s is the number of components that abstain totally.

The more general case of abstentions allows components to abstain in some statements, accept some others and reject the rest. Therefore, for each statement there is a different number of components abstaining. Let, for statement a_j, s components abstain. Then to decide if a_j belongs to $\mathcal{A}_{k/n}$ or not, we look at whether it belongs to $\mathcal{A}_{k(s)/n-s}$ or not. Where k_s is defined to be $k_s=[k_s']$ for $k\geq(n+1)/2$ and $k_s=[[k_s'']]$ for $k\geq(n+1)/2$ (recall Chapter 13 on Abstentions). Then we know that Conditions 1, 2 are satisfied and Conditions 3 and L are almost satisfied (almost fair abstention).

The Ordering Algorithm as discussed above takes the following form:

Given A, $\mathbf{x}$, $|\mathcal{A}_i|\geq\alpha$, φ symmetric k/n, we wish to find $B\subseteq A$ so that $||B|-\beta|$ is minimum and $|B|\geq\beta$ and so that the algorithm to find B is coherent.

Ordering Algorithm

STEP 1: Check if there are α-plets of statements in A so that their conjunction is not i-false. If no such α-plet exists STOP: A or α must be modified.

STEP 2: Find the outcome sets of the Voting Situations

$[A, k/n, \mathbf{x}, |\mathbb{A}_i| \geq \alpha]$ for $k=1,\ldots,n$. These are $\mathbb{A}_{k/n}$ for $k=1,\ldots,n$. In case of abstention use $\mathbb{A}_{k(s)/n-s}$ instead of $\mathbb{A}_{k/n}$ where $k_s=[k_s']$ for $k \geq (n+1)/2$ and $k_s=[[k_s'']]$ for $k \leq (n+1)/2$.

STEP 3: Determine the choice set B.

- If $\mathbb{A}_{1/n}=\emptyset$ STOP: "All components abstain on all issues"
- If $\mathbb{A}_{1/n} \not\subseteq \emptyset$ but $|\mathbb{A}_{1/n}|<\beta$ then we cannot find B so that $|B| \geq \beta$. STOP: "$|B|<\beta$"
- If $|\mathbb{A}_{1/n}| \geq \beta$ find j so that $|\mathbb{A}_{j/n}| \geq \beta$ but $|\mathbb{A}_{(j+1)/n}|<\beta$.
    - If such a j exists, set $B=\mathbb{A}_{j/n}$ STOP.
    - If no such j exists set $B=\mathbb{A}_{n/n}$ STOP since $|\mathbb{A}_{n/n}| \geq \beta$.

Discussion

1. The algorithm is complete as it stops in a finite number of steps.

2. The algorithm is coherent as the following arguments show:

Let $a_j \notin \mathbb{A}_i$ but $a_j \in \mathbb{A}_{k/n}$ and let there be s abstentions for statement a_j. This means that $a_j \in \mathbb{A}_{k(s)/n-s}$. Let now component i change his opinion and pass a_j so that $a_j \in \mathbb{A}_i$, and all other components vote the same way for a_j, while for all other statements in A except a_j, all components keep their opinions unchanged. We distinguish two cases,

Case 1: Component i was abstaining on a_j before.

It follows that since a_j attained at least k_s votes before, it will obtain now k_s+1 votes out of n-(s-1) voting components since s-1 components abstain now for a_j. Therefore,

$a_j \in \mathcal{A}_{k(s)+1/n-s+1}$ and thus $a_j \in \mathcal{A}_{k(s-1)/n-s+1}$ since $k_s+1 \geq k_{s-1}$ because $0 \leq k_{s-1}-k_s \leq 1$. But $\mathcal{A}_{k(s-1)/n-s+1}$ corresponds to $\mathcal{A}_{k/n}$ when there are s-1 abstentions and therefore $a_j \in \mathcal{A}_{k/n}$ again in the new situation.

Case 2: Component i was rejecting a_j before.

It follows that now a_j gets k_s+1 votes in n-s components that vote for a_j since s abstain again. Therefore, $a_j \in \mathcal{A}_{k(s)/n-s}$, which corresponds to $\mathcal{A}_{k/n}$ and then once more $a_j \in \mathcal{A}_{k/n}$.

3. A generalization of the Order Hierarchy described above can be made by replacing the k/n symmetric structures in the sequence of Voting Situations by a sequence of coherent structures with n components each, $\{\varphi_k\}$, $k=1,\ldots,K$, so that φ_K is n/n and φ_1 is 1/n and each min path of φ_i is a subset of some min path of φ_{i+1}. This sequence of v.s.'s will give outcome sets so that $\mathcal{A}_{\varphi(i)} \supseteq \mathcal{A}_{\varphi(i+1)}$ for all i, which implies an order on statements in A as desired.

Observe also that if a sequence of structures with n components each, $\varphi_1,\ldots,\varphi_K$, implies an order on the statements in A, then so does the sequence that results when the same contraction is applied to each φ_k. However, when the structures used in the sequence of v.s.'s is not symmetric, the problem of abstentions is difficult to deal with.

Example 14.1

Consider the elections for the members of the board of an association. The components are asked to vote for exactly α candidates and abstentions are total abstentions. The votes are counted and the sequence of sets B_0, B_1, ... is formed so that B_i *contains those candidates that obtain i number of*

votes. Here B must be chosen so that $|B|=\beta$ (the number of seats). To do this pick the β with the higher number of votes. Ties are broken by chance.

Example 14.2

This example is one where voting with weight appears. It is the type of voting used to elect the best songs in the European Song Competition. We will show that it is the same as the Ordering Algorithm.

Description:

There are 19 countries participating and there are 19 songs. Each country allocates 12 votes to the song it regards best, 10 votes to the second, 9 votes to the third,..., 2 votes to the tenth and 1 vote to the eleventh song.

Abstentions are not allowed except total abstentions. Each country, therefore, allocates a total of 58 votes (1+2+...+10+12=58). Since there are 19 countries (let there be no abstentions for simplicity) there are 19x58 =1102 votes casted in total. Also the maximum possible number of votes any song may receive is 12x19=228 votes if it happens to be considered the best by all 19 countries. Finally songs are ordered according to the number of votes they receive.

Let us give to the description above the form of an Ordering Algorithm:

A: the set of alternatives containing the 19 songs.

$\varphi_1,\ldots\varphi_{228}$ is a sequence of structures resulting from a sequence of k/n structures by contraction and omission of paths (OP) as described below.

$|A_j|=1$ for all components in the contracted sequence of structures $\varphi_1, \ldots \varphi_{228}$.

The sequence $\{\varphi_i\}$ $i=1,\ldots,228$ is constructed as follows,

First observe that each country can be expanded to:

one group of 12 "components"

one group of 10 "components"

one group of 9 "components"

. . .

one group of 1 "component"

This is a total of 58 "components" for each country and thus there are a total of 58x19=1102 "components" each voting for one song only.

Imagine now the sequence of structures {k/1102} for k=1,...,228 (since a song cannot receive more than 228 votes). All "components" within a group vote identically and therefore contraction applies (12 components, 10 components,...,1 component groups for each country). However, after contraction, paths containing contracted components (groups originally) of the same country must be omitted, because they are always in disagreement, since a country cannot consider the same song both say second and fifth in its choice.

The sequence of φ_i's thus constructed can be used for the Ordering Algorithm, because each path of φ_i is a subset of some path of φ_{i+1} for all i. To show this assertion consider the jth min path, P_{ij}, of φ_i. This path contains no disagreement groups, otherwise it would have been omitted. It is therefore, the result of contraction only from a min path of i components of the i/1102 structure that contains no disagreement groups. Call this min path Q_{ij}. But each min path of i/1102 is a

subset of some min path of $(i+1)/1102$. Therefore, pick a path, call it $Q_{(i+1),k}$ so that $Q_{i+1,k} \supseteq Q_{ij}$ and so that the extra component that $Q_{(i+1),k}$ contains is not in disagreement with the rest which are exactly the same as those of Q_{ij}. This is possible because a symmetric structure contains all possible combination of components as its min paths. Therefore, the contraction of $Q_{(i+1),k}$, call it $P_{(i+1),k}$, is not omitted and then it is a min path of φ_{i+1} and $P_{(i+1),k} \supseteq P_{ij}$ as required.

A variation of voting with weight, as described above, is possible if components (countries) are allowed to cast their 58 votes as they wish among the 19 songs. Then pure contraction of the groups of "components" within each country from the sequence $\{k/1102\}$; $k=1,\ldots,1102$ is enough to produce the sequence of structures needed to apply the Ordering Algorithm since there are no disagreement groups.

14.3 Logic Hierarchy

Suppose that we want to use the Ordering Algorithm to choose as usual B ($B \subseteq A$) so that $||B|-\beta|$ is minimum while $|B| \geq \beta$ where β is given. But now we also want the statements in B not to be i-false under conjunction. ($\mathbb{N}_{AND}[B]=\emptyset$).

Theorem 7.17 identifies the family of coherent structures so that $|B| \geq \beta$ and $\mathbb{N}_{AND}[B]=\emptyset$. But they are very restrictive. The question, therefore, is whether there is some solution without having to use a "terminating" structure from those of Theorem 7.17, that is closer to the components' wishes. But first what do we mean by the phrase "closer to the components' wishes"?

Suppose that we are given the set of alternatives A and a set N of n components, $N=\{1,\ldots,n\}$. Also let $N_1 \subseteq N$, $N_2 \subseteq N$, $|N_1|=n_1$, $|N_2|=n_2$, $A_1 \subseteq A$, $A_2 \subseteq A$, $\alpha_1 \geq 1$, $\alpha_2 \geq 1$, $\mathbf{x}_1$ be the vector of characteristic functions of components in N_1 and $\mathbf{x}_2$ for components in N_2.

Definition

The ordering of alternatives implied by the v.s.'s $[A_1, k/n, \mathbf{x}_1, \mathbb{A}_i \geq \alpha_1]$ for $k=1,\ldots,n_1$ is *closer to the components' wishes* than that implied by $[A_2, k/n, \mathbf{x}_2, \mathbb{A}_i \geq \alpha_2]$ for $k=1,\ldots,n_2$ if

(a) $n_1 > n_2$

or if

(b) $n_1=n_2$ but $|A_1|>|A_2|$

or if

(c) $n_1=n_2$ and $|A_1|=|A_2|$ but $\alpha_1<\alpha_2$

To justify the definition observe that: (a) The ordering implied when more people are participating is preferred. (b) In case the number of people participating is the same, then the less restricted set of alternatives is preferred. (c) If both the number of people and the number of statements in the set of alternatives is the same then the less restricted individual choice is preferred. We may also observe that a *lexicographic ordering* of the triplets $(n, A, \overline{\alpha})$ is implied by the definition above.

Going now back to our original aim, we want to search for a subset B of A so that $|B|\geq\beta$, with $||B|-\beta|$ minimum, where B is formed by the Ordering Algorithm for (n', A', α') but also $\mathbb{N}_{AND}[B]=\emptyset$ and so that (n', A', α') is lexicographically maximal (or closest to the components' wishes); and where we are given n, A, α and $n'\leq n$, $A'\subseteq A$, $\overline{\alpha}'\leq\overline{\alpha}$. The following algorithm performs such a search,

Logic Hierarchy

STEP 1: (Initialization)

Set $N_V=N$, $A_V=A$, $n_V=n$

Check if A contains identically false statements. If it does STOP: "Problem is impossible: A contains i-false statements"

(Note that always $\beta \leq |A_V| - 1$ and $|B| \leq |A_V|$.)

STEP 2: (Check Logical Consistency of A_V)

(a) If all β-plets of A_V are false under conjunction, then

- If $A_V = A$, STOP: "Problem is impossible: No consistent β-plets can be formed in A."
- If $A_V \subset A$, GO TO STEP 7.

(Note that if $\beta = 1$ and $A_V \subset A$ then this case does not occur since we would have stopped at STEP 1.)

(b) If there are β-plets that are not i-false under conjunction, then continue to STEP 3.

STEP 3: (Initialize α)

Set $\alpha = 1$

STEP 4: (Ordering Algorithm)

Apply the Ordering Algorithm with v.s.'s: $[A_V, k/n_V, \mathbf{x}_V, |\mathcal{A}_i| \geq \alpha]$ for $k = 1, \ldots, n_V$, with $|B| \geq \beta$ and $||B| - \beta|$ minimum. The Ordering Algorithm stops at the following possibilities:

(a) If all α-plets in A_V are i-false under conjunction, then A_V must be modified. GO TO STEP 5.

(b) If all components totally abstain (i.e. if $\mathcal{A}_{1/n} = \emptyset$):

- If $\alpha = 1$ STOP: "No solution is possible since all components abstain on all issues".

(Note that even if A_V is modified (restricted) it is assumed that components will still abstain. This is reasonable since when a component abstains in a given situation he will also abstain if

the situation is more restrictive.)

- If $\alpha>1$ then GO TO STEP 5.

(c) If $|B|<\beta$: Set $\alpha=\alpha+1$ and GO TO STEP 4.

(Note α never exceeds $|A_V|-1$ since when $\alpha=|A_V|-1$ ($\beta\leq|A_V|-1$) then the Ordering Algorithm for K=1 (1/n structure) will give at least β statements in the outcome set $\mathcal{A}_{1/n}$. Therefore, $|B|\geq\beta$ and Case (c) would not occur).

(d) If $|B|\geq\beta$, GO TO STEP 6

(Let in this case $B=\mathcal{A}_{m/n}$)

STEP 5: (Modify A_V)

(a) If we come from Step 4(a):

(1) If $\beta=|A_V|-1$ and $|A_V|\geq 3$, then no modification of A_V is possible: GO TO STEP 7.

(Note that if a statement is eliminated from A_V, then $b=|A_V(\text{new})|$ and then we require $B=A_V(\text{new})$. But if we come to Step 5 from 4(a) we know that all α-plets in A_V are i-false and since $\alpha\leq|A_V|-1$, it follows that all β-plets in A_V are i-false and therefore B cannot be found.)

(2) If $|A_V|=2$. STOP: "A contains i-false statements"

(In fact this case will not occur because if we come from Step 4(a) we know that α-plets in $|A|$ are i=false. But $\alpha=1$ since $|A_V|=2$ and $\alpha\leq|A_V|-1$. Thus single statements of A_V, and hence A, are i-false. But this situation would have been exposed in Step 1.)

(3) Otherwise GO TO 5(c).

(b) If we come to Step 5 from Step 4(b):

(1) If $|A_V|=1$. No modification of A_V is possible. GO

TO STEP 7.

(2) Otherwise GO TO 5(c).

(c) Elimination procedure:

(1) If $\beta = |A_V| - 1$ then no modification of A_V is possible: GO TO STEP 7.

(2) If $\beta \leq |A_V| - 2$, then eliminate one statement from A_V:

Look at the order assigned to statements by the Ordering Algorithm with the series of v.s.'s $[A, k/n_V, \mathbf{x}, |\mathcal{A}_i| \geq 1]$ for $k = 1, \ldots, n_V$. Pick the minimum j so that $A_V \cap B_{j/n(v)} \neq \emptyset$ where $B_{r/n(v)}(A) \equiv \mathcal{A}_{r/n(v)}(A) - \mathcal{A}_{r+1/n(v)}(A)$ for $r = 1, \ldots, n_V$ and $B_{0/n(v)}(A) = A - \mathcal{A}_{1/n(v)}(A)$. Call this minimum j, q. Pick an element, say a_1, from $A_V \cap B_{q/n(v)}(A)$, which is the set of statements in A_V that get the least number of votes when statements are ordered according to the v.s.'s described above. Call this element a_1 and eliminate it from A_V.

If there are more than one elements in $A_V \cap B_{q/n(v)}(A)$ from which to choose one for elimination from A_V, pick the one that has the highest chance of creating an i-false statement. To do the latter, check all β-plets of A_V that are not i-false under conjunction. Order ("Logic Order") the alternatives in A_V according to the number of not i-false β-plets in which they participate. Finally, eliminate that statement in $A_V \cap B_{q/n(v)}(A)$ call it a_1, that has the lowest "logic order". In case of a tie, break it by chance. Set $A_V = A_V - \{a_1\}$ and GO TO STEP 2.

STEP 6 (Check for Logic Consistency)

Construct all logically false statements by conjunction of statements in B. This is the set defined as $\mathbb{N}_{AND}[B]$. Observe that to arrive at this step $|B|\geq\beta$ and $||B|-\beta|$ is minimal.

(a) If $\mathbb{N}_{AND}[B]=\emptyset$ STOP: B is the answer.

(b) If $\mathbb{N}_{AND}[B]\neq\emptyset$

(1) if $|B|>\beta$. Let $\delta=|B|-\beta$ $(\delta\geq 1)$

Check if by eliminating $1, 2, \ldots, \delta$ statements from B in all possible combinations, the remaining statements in B (call the set B') have $\mathbb{N}_{AND}[B']=\emptyset$. Then B' is an answer. STOP. Otherwise GO TO 6(c).

(2) If $|B|=\beta$ GO TO 6(c)

(c)

(1) If $m\geq 2$ set $B=\mathbb{A}_{m-1/n(v)}$ and GO TO STEP 6.

(2) If $m=1$ then $\mathbb{A}_{1/n}$ does not contain β statements that are logically consistent:

(2.1) If $\alpha\leq|A_V|-2$, then

set $\alpha=\alpha+1$ and GO TO STEP 4.

(2.2) If $\alpha=|A_V|-1$, then STOP: "The components are not NAND consistent".

STEP 7 (Modify n_V)

(This step is never used when $\beta=1$ because an answer will be found by varying α and A_V as we know by theory of Chapter 7.)

Eliminate one component at random from N_V and call it q. Set $N_V=N_V-\{q\}$, $n_V=n_V-1$, $A_V=A$ and GO TO STEP 3.

Remark 1 (Termination of the Algorithm)

Let

$n_f \equiv \max_{\rho,\epsilon}\{(A-\beta+1)\rho+\epsilon\}$ so that $1+(\beta-1)\rho \leq \epsilon \leq A-\beta$

and $n_f \leq n$. Let the maximizing ρ, ϵ be called ρ_f, ϵ_f.

By Theorem 7.17 we know that the symmetric structure k_f/n_f with $k_f=(\alpha-\beta+1)\rho_f+\epsilon_f$ and $\alpha=A-1$ will have $|\mathcal{A}_{k(f)/n(f)}(A)| \geq \beta$ and the structure will also belong to $\mathcal{S}_A$. Thus when $\alpha=A-1$, the outcome set $\mathcal{A}_{k(f)/n(f)}$ will have all possible AND conjunctions of its statements consistent. Further, n_f is the maximum possible that satisfies $n_f \leq n$. It follows that repetitive use of Step 7 will lead to $n_v=n_f$. In this case the Ordering Algorithm in Step 4 will find for $k=k_f$ that $|B| \geq \beta$ and going to Step 6 will terminate at 6(a) if it does not terminate earlier.

Theorem 7.17 requires that $|\mathcal{A}_i|=\alpha=A-1$. If, however, $|\mathcal{A}_i| \geq \alpha=A-1$, then some components will have $|\mathcal{A}_i|=A-1$ and some $|\mathcal{A}_i|=A$. Thus, it is possible that either $|\mathcal{A}_\varphi|=A-1$ or $|\mathcal{A}_\varphi|=A$. In the first case since $\varphi \in \mathcal{S}_A$ $\mathcal{N}_{AND}[\mathcal{A}_\varphi]=\emptyset$ and we terminate at Step 6(a) $(B=\mathcal{A}_\varphi)$. In the latter case where $|\mathcal{A}_\varphi|=A$ the fact that at least for one component j, $|\mathcal{A}_j|=A$, presumes that $\mathcal{N}_{AND}[A]=\emptyset$ otherwise j would not be NAND consistent. Therefore, once again we terminate in Step 6(a).

Remark 2

The Logic Hierarchy is coherent.

Suppose that $\mathcal{A}_i(N_v, A_v, \alpha_v)$ denotes the passing set of component i when the set of alternatives A is restricted to A_v; when the restriction $|\mathcal{A}_i| \geq \alpha_v$ is imposed; and when the set of participating components, N, is restricted to

N_V.

We will say that the Logic Hierarchy (L.H.) is a coherent algorithm, if whenever $a_j \in B$ and $a_j \notin \mathcal{A}_i(N_1, A_1, \alpha_1)$, then if component i changes opinion so that $a_j \in \mathcal{A}_i(N_1, A_1, \alpha_1)$, while all other components do not change opinion with respect to this or other alternatives and any other Voting Situation (N_V, A_V, α_V) in the L.H., then $a_j \in B'$, where B' is the new outcome set of the L.H.

To prove that the L.H. is coherent, suppose that $a_j \in B$ when $a_j \notin \mathcal{A}_i(N_1, A_1, \alpha_1)$ and let component i decide instead that $a_j \in \mathcal{A}_i(N_1, A_1, \alpha_1)$. We have to show that still $a_j \in B'$ where B' is the new outcome set of the L.H..

There are two cases,

Case 1

(N_i, A_1, α_1) is not along the path of search of the L.H.. Then nothing changes: If the algorithm is retraced from the beginning, we will end with the same outcome set i.e. $B'=B$. Therefore, $a_j \in B'$ since by assumption $a_j \in B$.

Case 2

Let (N_1, A_1, α_1) is along the path of search of the L.H.,

Case 2.1

Let $\alpha_1 \geq 2$.

The Ordering Algorithm (Step 4) is coherent. Therefore, if a_j received m votes before (i.e. $a_j \in \mathcal{A}_{m/n(1)}$ before), now it will

receive m+1 votes and thus $a_j \in \mathcal{A}_{m+1/n(1)}(\text{new}) \subseteq \mathcal{A}_{m/n(1)}(\text{new})$

Case 2.1.1

Suppose that B was chosen at this stage without any further modification of α_1 (i.e. $N_V = N_1$, $A_V = A_1$, $\alpha = \alpha_1$) and suppose that $B = \mathcal{A}_{q/n(1)}$ ($q \leq m$). Since $\mathcal{A}_{q/n(1)}(\text{new}) = \mathcal{A}_{q/n(1)}$, B' will also be chosen at this stage.

However, it is possible that B' may be different from B. This may occur if B' is chosen to be equal to $\mathcal{A}_{m+1/n}(\text{new})$ and this may occur only if $|\mathcal{A}_{m+1/n(1)}(\text{new})| = \beta$, whereas before $|\mathcal{A}_{m+1/n(1)}| < \beta$. But even in this case still $a_j \in B'$. If though, $\mathcal{A}_{m+1/n(1)}(\text{new})$ does not pass Step 6 (logic consistency test), then we go back from m+1 to m and pick $B' = \mathcal{A}_{m/n(1)} = B$ and once again $a_j \in B'$.

Case 2.1.2

Suppose that B was not chosen at this stage and further modification of α and perhaps A_1 or N_V was needed.

(a) The Ordering Algorithm did not provide enough answers ($|\mathcal{A}_{1/n(1)}| < \beta$) in which case the L.H. continued with modifying α_1 or A_1 or N_1. Then a_j will now enter $\mathcal{A}_{1/n(1)}(\text{new})$. Again either $\mathcal{A}_{1/n(1)}(\text{new}) < \beta$ and the algorithm continues as before thus leading to $B' = B$; or $\mathcal{A}_{1/n(1)}(\text{new}) = \beta$. In the latter case, if $\mathcal{A}_{1/n(1)}(\text{new})$ passes the logic consistency test of Step 6 then $B' = \mathcal{A}_{1/n(1)}(\text{new})$ and a_j belongs to B'. If, though, it does not pass the consistency test of Step 6, then the algorithm continues with modifying α_1 and perhaps A_1 or N_1 and

continues along the same path as before, thus leading to $B'=B$.

(b) The Ordering Algorithm did provide enough answers ($\mathcal{A}_{1/n(1)} \geq \beta$) but $\mathcal{A}_{1/n(1)}$ did not contain logically consistent β-plets.

Now $\mathcal{A}_{1/n(1)}$(new) will contain an extra element: a_j and hence,

- If $\mathcal{A}_{1/n(1)}$(new) contains consistent β-plets, then B' will be chosen as a subset of $\mathcal{A}_{1/n(1)}$(new) and contains a_j. Otherwise the β-plets in $\mathcal{A}_{1/n(1)}$(new) are the same as those in $\mathcal{A}_{1/n(1)}$ and B' would be logically inconsistent as before.
- If $\mathcal{A}_{1/n(1)}$(new) does not contain any logically consistent β-plet then the L.H. continues as before and $B'=B$.

Note that even if the L.H. continues and modifies A_1, still the path of search of the L.H. will remain intact since in the present Case 2.1 we assumed $\alpha_1 \geq 2$ and because the order with which statements are eliminated in Step 5 is determined according to components wishes when $|\mathcal{A}_i| \geq 1$ irrespective of the v.s.'s of the algorithm. Thus $B'=B$.

Case 2.2

Let $\alpha_1=1$

- If B is chosen at this stage without any further modification of α_1 or A_1, then the same arguments hold as in Case 2.1
- If B is not chosen at this stage, then the L.H. will continue changing lexicographically (n_1, A_1, α_1). When it reaches Step 5 to modify A_1, we have to examine if there is any change in the order with which statements are eliminated.

Since $\alpha_1=1$, the order of elimination will not change except for a_j which moves from $\mathcal{A}_{m/n(1)}(A)$ to $\mathcal{A}_{m+1/n(1)}(A)$. All other sets $\mathcal{A}_{m-1/n(1)}$, $\mathcal{A}_{m-2/n(1)},\dots,\mathcal{A}_{1/n(1)}$ will not change. Further, the "logic order" as described in Step 5 will not change and therefore, the L.H. will continue on the same path as before. It follows that as a_j was not eliminated before $(a_j \in B)$, by retracing the algorithm and eliminating in the same order as before, a_j will again belong to B'.

Remark 3

If $\beta=1$, there is a k so that the k/n structure that assures answers and logic according to Theorem 7.12 for any given n. In this case Step 7 of the L.H. for diminishing n, will never be used.

Remark 4

Variations of the L.H. can be thought of by altering the lexicographic order to, say, (A, n, α) or by asking that $|B| \leq \beta$ or $|B|=\beta$ or by demanding that $|\mathcal{A}_i| \leq \alpha$ or $|\mathcal{A}_i|=\alpha$. Such changes though should be performed with caution as the L.H. needs appropriate readjustments.

Remark 5

As the L.H. evolves, new voting situations appear. It makes quite a difference on the way components vote, whether these voting situations imply revoting each time or whether information on all possible votings was gathered before starting the algorithm. This

may require the filling of a huge questionnaire, which in reality may prove impractical. Revoting overcomes this difficulty but of course creates another: How much information on how the L.H. has proceeded so far should components have when they are asked to revote; because we expect components to adapt their voting habits (or strategy) according to the information they have or expect to collect.

Remark 6

In Step 5 where A_V is modified, statements are eliminated according first to the order implied by the Ordering Algorithm when $|\mathcal{A}_i(A)| \geq \alpha$ for $\alpha=1$ and second, in case of ties, according to their "logic order". We choose $\alpha=1$ instead of $\alpha=2$ or $\alpha=3$, etc. because in this way the passing set of each component i is least restricted and thus a more genuine opinion (and implied ordering) is revealed.

Example 14.3

Logic Hierarchy can be applied to voting where preference profiles are the issues. In particular let $C \equiv \{c_1, \ldots, c_m\}$ be a set of alternatives. Each component i is able to order c_j's according to his preferences. Define now statements,

a_{ij}="c_i is preferred or indifferent to c_j" for each pair i, j

There are $\binom{m}{2}$ such statements. The set A will contain these statements and negations of them. Then A contains $2\binom{m}{2}$ statements. Each component is asked to pass α or more statements.

Using the L.H. we want to find B so that $|B|=\beta=\binom{m}{2}$.

This will direct the L.H. towards determining a complete

ordering. If β is chosen smaller so that $|B| \geq \beta$ the aim will be to find a partial ordering. As the L.H. proceeds logical cycles are broken by elimination of statements and if necessary by elimination of components.

14.4 Other Hierarchies

To further demonstrate the wideness of the concept of hierarchies of coherent structures, some well known algorithms are presented in this section in the form of hierarchies of coherent structures.

14.4.1 Simple Electoral Ordering

The Simple Electoral Ordering is a particular case of the Ordering Algorithm. It uses the sequence of Voting Situations:
$[\mathcal{A},\ (jK)/n,\ \mathbf{x},\ |\mathcal{A}_i|=1]$ for $j=1,\ldots,[n/K]$ to obtain the sequence of outcome sets, $\mathcal{A}_{K/n} \supseteq \mathcal{A}_{2K/n} \supseteq \ldots \supseteq \mathcal{A}_{[n/K]K/n}$.
Let now $a_q \in \mathcal{A}$. If there is $j \geq 1$ so that $a_q \in \mathcal{A}_{jK/n}$ but $a_q \notin \mathcal{A}_{(j+1)K/n}$, then we say that $a_q \in B_j$. If there is no such $j \geq 1$ then $a_q \in B_0$.
In elections in an area where the quota is defined to be k (usually k is the integer part of the division of the number of votes by the number of seats or the number of seats plus one) the elements of B_0 get zero seats each; those of B_1 get 1 seat each...; those of B_j get j seats each etc.
The algorithm is obviously coherent since it is a subcase of the Ordering Algorithm. Nevertheless, remainders of votes may exist and not all seats be allocated.

14.4.2 D' Hondt System of Elections

This system is equivalent to gradually diminishing the quota required to obtain a seat until all seats, S in number, are covered. First the votes n_i that party i obtains is successively divided by $j=1, 2, 3, \ldots$ to obtain numbers $c_{ij}=[n_i/j]$. These c_{ij}'s are placed in descending order for all i, j to form the sequence of numbers $\{y_1\}$. Then the steps of the algorithm are as follows:

STEP 1: $I=1$

STEP 2: $k=y_I$

STEP 3: Apply the Simple Electoral Algorithm,

$[A, jk/n, \mathbf{x}, 1]$ for $j=1, \ldots, [k/n]$

STEP 4: Allocation of seats:

If $a_q \in B_j$ set $S_q=j$

STEP 5:

If $\Sigma_q S_q=S$, then STOP: Party a_q gets S_q seats $q=1, 2, \ldots$

If $\Sigma_q S_q<S$, then set $I=I+1$ and GO TO STEP 2.

The Algorithm stops when exactly S seats are allocated. The final quota is of course given by the the final value of k.

14.4.3 Elections by Repeated Voting and Elimination

Suppose there are $|A|=A$ candidates and β seats

$(A \geq \beta)$. Let $B_{K/n} \equiv \mathcal{A}_{K/n} - \mathcal{A}_{K+1/n}$ i.e. $B_{K/n}$ is the set of those candidates that receive exactly K votes. The algorithm works in two steps:

STEP 1:
Apply the Ordering Algorithm, $[A, K/n, \mathbf{x}, |\mathcal{A}_i| \geq \alpha]$ for $K=1,\ldots,n$. Find the min K, say K′, so that $B_{K/n} \not\subseteq \emptyset$. Then:
- If $A - |B_{K'/n}| > \beta$, GO TO STEP 2
- If $A - |B_{K'/n}| \leq \beta$, then elect the candidates in $A - B_{K'/n}$ plus another $\beta - (A - |B_{K'/n}|)$ candidates by chance from $B_{K'/n}$ for a total of β seats. STOP.

STEP 2:
Set $A(\text{new}) = A(\text{old}) - B_{K'/n}$ and GO TO STEP 1.

A variation of this algorithm is to change Step 1 so that all candidates are eliminated except the best $\beta+1$. This is for example the case in elections for mayor in Greece or the presidential elections in France, where in the second voting only the two most powerful candidates participate.

Doubtless there are myriads of other algorithms used in real life as they seem to apply best to each particular situation. It is not our purpose to describe them all. As we explained in the beginning of this chapter, only some simple forms of wide application were to be presented with the purpose of showing the similarity in the underlying principles and their relation to coherent structures.

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